Convergence of utility indifference prices to the superreplication price*

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Abstract

A discrete-time financial market model is considered with a sequence of investors whose preferences are described by concave strictly increasing functions defined on the positive axis. Under suitable conditions we show that, whenever their absolute risk-aversion tends to infinity, the respective utility indifference prices of a bounded contingent claim converge to its superreplication price.

Keywords: utility indifference price, superreplication price, convergence, utility maximization, risk aversion.

1 Introduction

In the present paper a sequence of investors is considered. Preferences of investor n are expressed via the choice of his or her concave strictly increasing utility function U_n. We treat the case $\mathrm{dom}(U_n) = (0, \infty)$.

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The utility indifference price (also called Hodges-Neuberger price or reservation price) for the seller of a contingent claim has been introduced by Hodges and Neuberger (1989). It is the minimal amount a seller should add to his or her initial wealth so as to reach an expected utility when delivering the claim which is greater than or equal to the one he or she would have obtained trading in the basic assets only. The superreplication price is a utility free concept. It is the minimal initial wealth needed for hedging the claim without risk.

We show that (under appropriate technical conditions) the utility indifference prices of a bounded claim converge to its superreplication price when the absolute risk-aversion $-U_n''/U_n'$ of the respective agents tends to infinity.

Up to now, this result was known essentially for exponential utility functions. See, among others and in various contexts, El Karoui and Rouge (2000), Bouchard (2000), Bouchard *et al.* (2001), Collin-Dufresne and Hugonnier (2004) and Delbaen *et al.* (2002). Note that the particular techniques for exponential functions can not be used for general utility functions. Moreover, we do not rely on the duality machinery and treat directly the primal problem.

Convergence results can also be found in Jouini and Kallal (2001), for another concept of utility price in a finite probability space and in Carassus and Rásonyi (2005a) for reservation and Davis prices (in the same framework as the present paper). More precisely, the convergence of those prices was shown when U_n tend to some limiting utility function U_∞. This is connected with the main result of this paper noting that the superreplication price can be considered as the utility indifference price for the function

$$U_\infty(y) := -\infty,\ y < x, \quad U_\infty(y) := 0,\ y \geq x,$$

where x is the agent's initial capital, see section 3 for details. But we can neither apply directly the results of Carassus and Rásonyi (2005a) nor the same techniques since they are based on smoothness of U_∞.

2 Definitions, assumptions and results

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ be a discrete-time filtered probability space with time horizon $T \in \mathbb{N}$. We assume that $\mathcal{F}_0$ coincides with the family of P-zero sets. Let $\{S_t,\ 0 \leq t \leq T\}$ be a d-dimensional adapted process representing the discounted (by some numéraire) price of d securities in a given economy. The notation $\Delta S_t := S_t - S_{t-1}$ will often be used. Denote by $D_t(\omega)$ the smallest affine hyperplane containing the support of the (regular) conditional distribution of ΔS_t with respect to $\mathcal{F}_{t-1}$, see Proposition 8.1 of Rásonyi and Stettner (2005a) for more information about the random set D_t.

Trading strategies are given by d-dimensional processes $\{\phi_t,\ 1 \leq t \leq T\}$ which are supposed to be predictable (*i.e.* ϕ_t is $\mathcal{F}_{t-1}$-measurable). The class of all such strategies is denoted by Φ. Denote by L^∞, L^∞_+ the sets of bounded, nonnegative bounded random variables, respectively, equipped with the supremum norm $\|\cdot\|_\infty$. Trading is assumed to be self-financing, so the value process of a portfolio $\phi \in \Phi$ is

$$V_t^{z,\phi} := z + \sum_{j=1}^{t} \langle \phi_j, \Delta S_j \rangle,$$

where z is the initial capital of the agent in consideration and $\langle \cdot, \cdot \rangle$ denotes scalar product in $\mathbb{R}^d$.

The following absence of arbitrage condition is standard:

$$(NA):\ \forall \phi \in \Phi\ (V_T^{0,\phi} \geq 0 \text{ a.s.} \Rightarrow V_T^{0,\phi} = 0 \text{ a.s.}).$$

However, we need to assume a certain strengthening of the above concept hence an alternative characterization is provided in the Proposition below. Let Ξ_t denote the set of $\mathcal{F}_t$-measurable d-dimensional random variables,

$$\tilde{\Xi}_t := \{\xi \in \Xi_t :\ \xi \in D_{t+1} \text{ a.s.},\ |\xi| = 1 \text{ on } \{D_{t+1} \neq \{0\}\}\}.$$

Proposition 2.1 *(NA) holds iff there exist $\mathcal{F}_t$-measurable random variables $\beta_t,\ 0 \leq t \leq T-1$ such that*

$$\operatorname*{ess.\,inf}_{\xi \in \tilde{\Xi}_t} P(\langle \xi, \Delta S_{t+1} \rangle < -\beta_t | \mathcal{F}_t) > 0 \ \textit{a.s. on}\ \{D_{t+1} \neq \{0\}\}. \tag{1}$$

Proof. The direction $(NA) \Rightarrow (1)$ is Proposition 3.3 of Rásonyi and Stettner (2005a). The other direction is clear from the implication $(g) \Rightarrow (a)$ in Theorem 3 of Jacod and Shiryaev (1998). □

The following condition is called "uniform no-arbitrage" and was introduced by Schäl (2000).

Assumption 2.2 There exists a constant $\beta > 0$ such that for $0 \leq t \leq T-1$

$$\operatorname*{ess.\,inf}_{\xi \in \tilde{\Xi}_t} P(\langle \xi, \Delta S_{t+1} \rangle < -\beta | \mathcal{F}_t) > 0 \text{ a.s. on } \{D_{t+1} \neq \{0\}\}.$$

Let $G \in L^\infty_+$ be a random variable which will be interpreted as the payoff of some derivative security at time T. Now the concept of superreplication price is formally introduced as the minimal initial wealth needed for hedging without risk the given contingent claim:

$$\pi(G) := \inf\{z \in \mathbb{R} : \ V_T^{z,\phi} \geq G \text{ for some } \phi \in \Phi\}.$$

We refer to Karatzas and Cvitanić (1993), El Karoui and Quenez (1995), Kramkov (1996) and Föllmer and Kabanov (1998) for more information about this notion.

We go on incorporating a sequence of agents in our model with concave utility functions U_n. The functions r_n below express the absolute risk-aversion of the respective agents.

Assumption 2.3 Suppose that $U_n : (0,\infty) \to \mathbb{R},\ n \in \mathbb{N}$ is a sequence of concave strictly increasing twice continuously differentiable functions such that

$$\forall x \in (0,\infty) \quad r_n(x) := -\frac{U_n''(x)}{U_n'(x)} \to \infty,\ n \to \infty.$$

We extend each U_n to $[0,\infty)$ by continuity.

Example 2.4 Typical examples are the sequences $U_n(x) = -e^{-\gamma_n x},\ x > 0$ where $0 < \gamma_n$ and $\gamma_n \to \infty$ or the utility functions with derivatives $U_n'(x) = e^{-\gamma_n x^2},\ x > 0$ where $0 < \gamma_n$ and $\gamma_n \to \infty$.

Define for each $x > \pi(G)$, the set $\mathcal{A}(G,x)$ of admissible strategies:

$$\mathcal{A}(G,x) := \{\phi \in \Phi : \ V_T^{x,\phi} \geq G \text{ a.s.}\}.$$

Define the supremum of expected utility at the terminal date when delivering claim G, starting from initial wealth $x \in (\pi(G),\infty)$:

$$u_n(G,x) := \sup_{\phi \in \mathcal{A}(G,x)} EU_n(V_T^{x,\phi} - G), \tag{2}$$

where the expectations exist if S is bounded and Assumption 2.2 holds, see Lemma 3.2 below.

Remark 2.5 It would also be possible to extend U_n on the negative half-line as $-\infty$. In this case one may work without the admissibility condition on strategies and with arbitrary initial wealth.

Definition 2.6 The utility indifference price $p_n(G,x)$ is defined as

$$p_n(G,x) = \inf\{z \in \mathbb{R} : u_n(G,x+z) \geq u_n(0,x)\}.$$

We wish to find conditions on S and U_n which guarantee that $p_n(G,x)$ tends to $\pi(G)$ whenever Assumption 2.3 holds. Our main result is the following Theorem, see also Remark 3.6 for possible generalizations.

Theorem 2.7 *Suppose that $x \in (0,\infty)$, S is bounded, Assumptions 2.2 and 2.3 hold. Then the utility indifference prices $p_n(G,x)$ are well-defined and converge to $\pi(G)$ as $n \to \infty$.*

3 Proof of the main result

Lemma 3.1 *Let $x > \pi(G)$. Suppose that S is bounded and Assumption 2.2 holds. Take any strategy $\phi \in \mathcal{A}(G,x)$ satisfying $\phi_t \in D_t$, $1 \le t \le T-1$. There exist increasing functions $M_t(x) \ge 0$ such that*

$$V_t^{x,\phi} \le M_t(x).$$

Proof. For $t = 0$ take $M_0(x) := x$. Suppose that the statement has been shown up to $t-1$. We claim that

$$|\phi_t| \le \frac{V_{t-1}^{x,\phi}}{\beta}. \tag{3}$$

Indeed, define

$$A := \left\{ |\phi_t| > \frac{V_{t-1}^{x,\phi}}{\beta} \right\} \in \mathcal{F}_{t-1}, \;\; B := \left\{ \langle \frac{\phi_t}{|\phi_t|}, \Delta S_t \rangle < -\beta \right\}.$$

Clearly, $\{V_t^{x,\phi} < 0\} \supset A \cap B$ and

$$P(A \cap B) = E[E[I_{A\cap B}|\mathcal{F}_{t-1}]] = E[I_A[E(I_B|\mathcal{F}_{t-1})]].$$

By Assumption 2.2, $P(B|\mathcal{F}_{t-1}) > 0$, thus $P(A) > 0$ would imply that $P(V_t^{x,\phi} < 0) > 0$. But as $V_T^{x,\phi} \ge G \ge 0$ a.s., the no-arbitrage condition (NA) implies that $V_t^{x,\phi} \ge 0$ a.s. for all t. This contradiction shows that (3) holds.

Thus by the induction hypothesis

$$V_t^{x,\phi} \le M_{t-1}(x) + \|\Delta S_t\|_\infty M_{t-1}(x)/\beta =: M_t(x),$$

which defines a suitable $M_t(x)$. □

Lemma 3.2 *Let $x > \pi(G)$. If S is bounded and Assumption 2.2 holds then $u_n(G,x)$ is well-defined, finite and*

$$u_n(G,x) = \sup_{\phi \in \mathcal{A}(G,x),\ \phi_t \in D_t} EU_n(V_T^{x,\phi} - G).$$

Proof. Take some strategy $\tilde{\phi} \in \mathcal{A}(G,x)$ such that $V_T^{x,\tilde{\phi}} \geq G + \varepsilon$ for some $\varepsilon > 0$. Then

$$u_n(G,x) \geq U_n(\varepsilon) > -\infty.$$

Let $\phi \in \mathcal{A}(G,x)$ and $\hat{\phi}_t(\omega)$ be the orthogonal projection of $\phi_t(\omega)$ on $D_t(\omega)$. From Lemma 3.1,

$$U_n(V_T^{x,\hat{\phi}} - G) \leq U_n(M_T(x)).$$

By definition of D_t,

$$\langle \phi_t, \Delta S_t \rangle = \langle \hat{\phi}_t, \Delta S_t \rangle \text{ a.s.}$$

and thus

$$u_n(G,x) = \sup_{\phi \in \mathcal{A}(G,x)} EU_n(V_T^{x,\phi} - G) = \sup_{\phi \in \mathcal{A}(G,x)} EU_n(V_T^{x,\hat{\phi}} - G) < \infty,$$

and the statements are proved. □

Denote by L^0 the set of all real-valued random variables on $(\Omega, \mathcal{F}, P)$ equipped with the topology of convergence in probability. The notation L^0_+ stands for the set of nonnegative random variables. Define for $z \in \mathbb{R}$

$$K_z := \{V_T^{z,\phi} : \ \phi \in \Phi\}.$$

We recall the following fundamental fact, see Kabanov and Stricker (2001) or Schachermayer (1992) for a proof.

Theorem 3.3 *Under (NA), the set $K_z - L^0_+$ is closed in probability.* □

Lemma 3.4 *Let $B \in L^0$ such that $B \notin K_z - L^0_+$. Then there exists $\varepsilon > 0$ such that*

$$\inf_{\theta \in K_z} P(\theta \leq B - \varepsilon) \geq \varepsilon.$$

Proof. Suppose that the statement is false. Then for all n there is $\theta_n \in K_z$ such that

$$P(\theta_n \leq B - 1/n) \leq 1/n,$$

hence for $\kappa_n := [\theta_n - (B - 1/n)] I_{\{\theta_n > B - 1/n\}} \in L^0_+$:

$$P(\theta_n - \kappa_n = B - 1/n) \geq 1 - 1/n.$$

This implies $\theta_n - \kappa_n \to B$ in probability, hence $B \in \overline{K_z - L^0_+} = K_z - L^0_+$, a contradiction. □

Lemma 3.5 *Suppose that U_n, $n \in \mathbb{N}$ satisfy Assumption 2.3 as well as*

$$\forall n \in \mathbb{N} \quad U_n(x) = 0, \quad U_n'(x) = 1,$$

for some fixed $x \in (0, \infty)$. Then

$$\forall y < x \quad U_n(y) \to -\infty,\ n \to \infty, \quad \forall y \geq x \quad U_n(y) \to 0,\ n \to \infty.$$

Proof. First take $y < x$. As U_n' is decreasing, $U_n'(u) \geq U_n'(x) = 1$, for $u \leq x$, hence $r_n(u) \leq -U_n''(u)$. Necessarily

$$U_n'(y) = U_n'(x) - \int_y^x U_n''(u)du \geq 1 + \int_y^x r_n(u)du \to \infty,$$

as $n \to \infty$, by the Fatou-lemma. Also

$$U_n(y) = U_n(x) - \int_y^x U_n'(u)du \to -\infty,$$

by the same reasoning, using the previous convergence observation.

Now for any $y > x$ we claim that $U_n'(y) \to 0$. If this were not the case, along a subsequence n_k, for all k

$$U_{n_k}'(y) \geq \alpha > 0.$$

Then by monotonicity $U_{n_k}'(u) \geq \alpha$, for all $u \leq y$, so $r_n(u) \to \infty$ implies that $-U_{n_k}''(u) \to \infty,\ k \to \infty,\ u \leq y$. Then necessarily

$$0 \leq U_{n_k}'(y) = U_{n_k}'(x) + \int_x^y U_{n_k}''(u)du = 1 + \int_x^y U_{n_k}''(u)du \to -\infty,$$

a contradiction proving the second assertion of this Lemma. □

Proof of Theorem 2.7. Fix $x > 0$. As we have already pointed out in Lemma 3.2, $u_n(G, x)$ is well-defined and finite.

Notice that Assumption 2.3 remains true if we replace each U_n by $\alpha_n U_n + \beta_n$ for some $\alpha_n > 0,\ \beta_n \in \mathbb{R}$. Also, the utility indifference prices defined by these new functions are the same as the ones defined by the original U_n. Hence by choosing $\alpha_n := 1/U_n'(x)$ and $\beta_n := -U_n(x)/U_n'(x)$, we may and will suppose that for all $n \in \mathbb{N}$

$$U_n(x) = 0, \quad U_n'(x) = 1. \tag{4}$$

Fix $\pi(G) < y < x + \pi(G)$. Then

$$x + G \notin K_y - L_+^0,$$

by the definition of the superreplication price. Take $0 < \varepsilon$ given by Lemma 3.4 appplied with $B := x + G$ and $z = y$. Notice that the function $M_T(\cdot)$ figuring in Lemma 3.1 does not depend on the particular choice of the strategy ϕ and hence can be chosen uniformly for all $\phi \in \mathcal{A}(G, y)$ such that $\phi_t \in D_t$ for all t. For such a ϕ, define the sets

$$A_\phi := \{\omega \in \Omega : \ V_T^{y,\phi}(\omega) \leq x + G(\omega) - \varepsilon\}.$$

It follows from Lemma 3.4 that $P(A_\phi) \geq \varepsilon$. We get

$$\begin{aligned} EU_n(V_T^{y,\phi} - G) &\leq EI_{A_\phi} U_n(x - \varepsilon) + EI_{A_\phi^C} U_n(M_T(y)) \\ &\leq P(A_\phi)U_n(x - \varepsilon) + U_n(M_T(y) + x)P(A_\phi^C) \\ &\leq \varepsilon U_n(x - \varepsilon) + U_n(M_T(y) + x). \end{aligned} \tag{5}$$

For the last inequality we used the fact that $U_n(x-\varepsilon) \leq U_n(x) = 0$ and that $U_n(z) \geq 0$ for all $z \geq x$. Thus, by Lemma 3.2

$$u_n(G, y) \leq \varepsilon U_n(x - \varepsilon) + U_n(M_T(y) + x) \to -\infty, \tag{6}$$

by Lemma 3.5.

We also see from the definition of $u_n(0, x)$ that

$$\liminf_{n\to\infty} u_n(0, x) \geq \liminf_{n\to\infty} U_n(x) = 0. \tag{7}$$

One may easily check that

$$p_n(G, x) \leq \pi(G). \tag{8}$$

Indeed, for any $\delta > 0$ we may take a strategy $\hat{\phi}(\delta) \in \mathcal{A}(G, \pi(G) + \delta)$ such that

$$V_T^{\pi(G)+\delta,\hat{\phi}(\delta)} \geq G.$$

Then, as U_n is non decreasing,

$$\begin{aligned} u_n(0, x) &\leq \sup_{\phi\in\mathcal{A}(0,x)} EU_n(V_T^{x+\pi(G)+\delta,\phi+\hat{\phi}(\delta)} - G) \\ &\leq \sup_{\phi\in\mathcal{A}(G,x+\pi(G)+\delta)} EU_n(V_T^{x+\pi(G)+\delta,\phi} - G) = u_n(G, x + \pi(G) + \delta), \end{aligned}$$

so by the definition of the utility indifference price $p_n(G, x) \leq \pi(G) + \delta$ and (8) follows by letting $\delta \to 0$.

Now it remains to prove

$$\liminf_{n\to\infty} p_n(G, x) \geq \pi(G). \tag{9}$$

Suppose that this fails, *i.e.* for some $x > \eta > 0$ and a subsequence n_k

$$p_{n_k}(G, x) \leq \pi(G) - \eta$$

holds, for all $k \in \mathbb{N}$. Again, by Definition 2.6,

$$u_{n_k}(G, x + \pi(G) - \eta) \geq u_{n_k}(0, x),$$

the left-hand side tends to $-\infty$ by (6) applied to $y = x + \pi(G) - \eta$ and the liminf of the right-hand side is nonnegative by (7), a contradiction proving (9) and hence the Theorem. □

Remark 3.6 It is possible to extend Theorem 2.7 to certain unbounded price processes and relax Assumption 2.2, too. Define $\mathcal{W}$ as the set of random variables with finite moments of all orders. Suppose $\Delta S_t \in \mathcal{W}$, $1/\beta_{t-1} \in \mathcal{W}$, $1 \leq t \leq T$ and Assumption 2.3. Then $p_n(G, x)$ tends to $\pi(G)$. Lemma 3.1 can be shown with random variables $M_t(x) \in \mathcal{W}$ instead of constants. Lemma 3.2 also follows easily. Then the same arguments work, just like in (5) we get

$$EU_n(V_T^{x,\phi} - G) \leq \varepsilon U_n(x - \varepsilon) + EI_{A_\phi^C} U_n(M_T(y) + x).$$

Here

$$I_{A_\phi^C} U_n(x + M_T(y)) \leq I_{A_\phi^C}[U_n(x) + M_T(y)U_n'(x)] \leq M_T(y),$$

and this is integrable (in fact, lies in $\mathcal{W}$), hence

$$u_n(G, y) \leq \varepsilon U_n(x - \varepsilon) + EM_T(y)$$

and $u_n(G, y) \to -\infty$ for $\pi(G) < y < \pi(G) + x$. The rest of the proof is identical.

4 Conclusion

It is well-known that exponential utility prices converge to superreplication price when agents' risk aversion tends to infinity. We have generalized this result to concave, strictly increasing, twice differentiable utility functions with domain $(0, \infty)$. What happens when the domain is the whole real axis? The same kind of results can be obtained but this requires different techniques and will be addressed elsewhere, see Carassus and Rásonyi (2005b).

Another natural question is the convergence of the optimal strategies to some superreplication strategy. This is false in general, see the example of Cheridito and Summer (2003). In the setting of the present paper, all considered strategies are superhedging strategies by the definition of the admissible set $\mathcal{A}(G, x)$. In particular, the optimal strategies, which exist by Rásonyi and Stettner (2005b), are superreplicating ones.

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