Optimal strategies and utility-based prices converge when agents' preferences do

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Abstract

A discrete-time financial market model is considered with a sequence of investors whose preferences are described by utility functions U_n defined on the whole real line. Under suitable hypotheses, it is shown that whenever U_n tends to a utility function U_∞, the respective optimal strategies converge too. Under additional assumptions, we estimate the rate of convergence. We also establish the continuity of the Davis and Hodges-Neuberger prices with respect to changes in agents' preferences.

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1 Introduction

In the present article we are interested in the following question: does the convergence of agents' preferences entail the convergence of the respective optimal strategies? We assume that these preferences are described by means of utility functions, i.e. strictly concave, increasing functions $U_n,\ n \in \mathbb{N}$ converging to some utility function U_∞. In Jouini and Napp (2004) the case of a complete Brownian market model was studied, where investors' utility functions were defined on the positive axis. It was shown that the convergence of optimal strategies indeed takes place under appropriate conditions.

In this paper we focus on different classes of models and agents: discrete-time markets with finite time horizon and utility functions defined on the whole real line. Note that these financial market models are, unlike the ones in Jouini and Napp (2004), generically incomplete. The study of such markets is totally different and more involved than that of complete markets. Thus we have to make extra assumptions such that strong no arbitrage and bounded price processes (see section 2.1 for precise definitions). In section 3, we will give counter-examples which show why such assumptions are necessary.

Our main result is that the convergence of utility functions implies the convergence of the respective optimal strategies. Under stronger assumptions we also show that the convergence rate is the same in both cases.

In incomplete markets the choice of a suitable pricing rule is a fundamental issue. So we establish the convergence of two types of utility-based prices: Davis price (see Davis (1997)) and utility indifference price (see Hodges and Neuberger (1989)).

2 Model description and main results

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, P)$ be a discrete-time filtered probability space with time horizon $T \in \mathbb{N}$. We assume that $\mathcal{F}_0$ coincides with the family of P-zero sets.

2.1 Market description

Let $\{S_t,\ 0 \le t \le T\}$ be a d-dimensional adapted process representing the discounted – by some numéraire – price process of d securities in a given economy. The notation $\Delta S_t := S_t - S_{t-1}$ will often be used. Trading strategies are given by d-dimensional processes $\{\psi_t,\ 1 \le t \le T\}$ which are supposed to be predictable (i.e. ψ_t is $\mathcal{F}_{t-1}$-measurable). The class of all such strategies is denoted by Φ. Denote by L^∞, L^∞_+ the sets of bounded, nonnegative bounded random variables, respectively, equipped with the supremum norm $\|\cdot\|_\infty$.

Trading is assumed to be self-financing, so the value process of a portfolio $\psi \in \Phi$ is

$$V_t^{z,\psi} := z + \sum_{j=1}^{t} \langle \psi_j, \Delta S_j \rangle,$$

where z is the initial capital of the agent in consideration and $\langle \cdot, \cdot \rangle$ stands for the inner product in $\mathbb{R}^d$.

The following absence of arbitrage condition is standard:

$$(NA):\ \forall \psi \in \Phi\ (V_T^{0,\psi} \ge 0 \text{ a.s.} \Rightarrow V_T^{0,\psi} = 0 \text{ a.s.}).$$

However, we need to assume a certain strengthening of the above concept hence an alternative characterization of (NA) is provided in the Proposition below. Denote by $D_t(\omega)$ the smallest affine hyperplane containing the support of the (regular) conditional distribution of ΔS_t with respect to $\mathcal{F}_{t-1}$. We refer to Proposition 8.1 of Rásonyi and Stettner (2005) for more details about the (random) set D_t. Let Ξ_t denote the set of $\mathcal{F}_t$-measurable d-dimensional random variables,

$$\tilde{\Xi}_t := \{\xi \in \Xi_t\ :\ \xi \in D_{t+1} \text{ a.s. } |\xi| = 1 \text{ on } \{D_{t+1} \neq \{0\}\}\}.$$

Proposition 2.1 *(NA) holds if and only if there exist $\mathcal{F}_t$-measurable, strictly positive, random variables $\beta_t,\ 0 \le t \le T-1$ such that*

$$\text{ess.}\inf_{\xi \in \tilde{\Xi}_t} P(\langle \xi, \Delta S_{t+1} \rangle < -\beta_t | \mathcal{F}_t) > 0 \text{ a.s. } on\ \{D_{t+1} \neq \{0\}\}. \tag{1}$$

Proof. The direction $(NA) \Rightarrow (1)$ is Proposition 3.3 of Rásonyi and Stettner (2005). The other direction is clear from the implication $(g) \Rightarrow (a)$ of Theorem 3 in Jacod and Shiryaev (1998). □

We formulate a stronger concept of absence of arbitrage. Similar strengthenings appeared in Schäl (1999, 2000).

Assumption 2.2 There exist constants $\beta, \kappa > 0$ such that

$$\text{ess.} \inf_{\xi \in \tilde{\Xi}_t} P(\langle \xi, \Delta S_{t+1} \rangle < -\beta | \mathcal{F}_t) > \kappa \text{ a.s. on } \{D_{t+1} \neq \{0\}\}.$$

We show in section 3 that the problems of interest in this paper can be ill-posed if Assumption 2.2 is not satisfied.

The following technical assumption roughly says that there are no redundant assets, even conditionally. It is possible to work without it, see Remark 2.13.

Assumption 2.3 D_t is almost surely equal to $\mathbb{R}^d$, for all $1 \leq t \leq T$.

2.2 Agents' preferences

Introduce the notation $\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$. Consider a sequence of agents with preferences converging to some limiting preference.

Assumption 2.4 Suppose that $U_n : \mathbb{R} \to \mathbb{R},\ n \in \bar{\mathbb{N}}$ is a sequence of strictly concave and increasing continuously differentiable functions such that for all $x \in \mathbb{R}$

$$U_n(x) \to U_\infty(x), \quad n \to \infty.$$

Remark 2.5 Note that the above Assumption implies the uniform convergence of both U_n and U_n' on compacts, by p. 90 and p. 248 of Rockafellar (1970).

A further technical condition needs to be imposed.

Assumption 2.6 Assume that there exist $0 < \gamma < 1,\ \tilde{x} > 0$ such that for all $\lambda \geq 1$, $y \geq \tilde{x}$ and for all $n \in \bar{\mathbb{N}}$

$$U_n(\lambda y) \leq \lambda^\gamma U_n(y).$$

Remark 2.7 This assumption says that agents' utility functions satisfy a certain "uniform asymptotic elasticity" condition at $+\infty$, see Kramkov and Schachermayer (1999), Schachermayer (2001) and Remark 2.4 of Rásonyi and Stettner (2005) about this notion, compare also to property (P3) on p. 135 of Jouini and Napp (2004). Without some hypothesis of this kind there might not exist an optimal strategy, see section 7 of Rásonyi and Stettner (2005). All results of the present paper hold under a similar uniform asymptotic elasticity condition at $-\infty$ instead of $+\infty$.

In case we would like to estimate the rate of convergence, a strengthening of Assumption 2.4 is needed.

Assumption 2.8 The functions $U_n,\ n \in \bar{\mathbb{N}}$ are strictly concave, increasing and twice continuously differentiable. For all $N > 0$, the second derivative satisfies the bounds

$$\ell(N) \leq |U_n''(x)| \leq L(N),\ x \in [-N, N],\ n \in \bar{\mathbb{N}},$$

with constants $\ell(N),\ L(N) > 0$ and there exists a sequence of real numbers $g(n) \to 0,\ n \to \infty$ such that

$$|U_n(0) - U_\infty(0)| + \sup_{x\in[-N,N]} |U'_n(x) - U'_\infty(x)| \leq C(N)g(n),\ n \in \mathbb{N}, \tag{2}$$

where the $C(N)$ are suitable constants.

Remark 2.9 The condition on U''_n is a kind of "uniform strict concavity" property. Under Assumption 2.8 the inequality

$$|U_n(x) - U_\infty(x)| \leq |U_n(0) - U_\infty(0)| + \int_0^x |U'_n(y) - U'_\infty(y)|dy \tag{3}$$

shows that U_n tends to U_∞ uniformly on compacts with convergence speed $O(g(n))$. Note that if U_n tends to U_∞ uniformly on compacts with convergence speed $O(g(n))$ then (2) does not always hold true.

If we assume that U''_n converges to U''_∞ at the rate $g(n)$, $U''_\infty < 0$ and also that there exists some $x_0 \in \mathbb{R}$ such that $U_n(x_0)$ and $U'_n(x_0)$ converge respectively to $U_\infty(x_0)$ and $U'_\infty(x_0)$ at the rate $g(n)$, then one can prove (by an argument similar to (3)) that Assumption 2.8 holds.

Example 2.10 Typical examples are the sequences $U_n(x) = -e^{-\alpha_n x},\ x \in \mathbb{R},\ 0 < \alpha_n,\ n \in \bar{\mathbb{N}}$ where $\alpha_n \to \alpha_\infty$ at a given rate $O(g(n))$.

2.3 Optimization problems and convergence of optimal solutions

Fix any element $G \in L^\infty_+$ and define

$$u_n(G, z) := \sup_{\psi\in\Phi(U_n,G,z)} EU_n(V_T^{z,\psi} - G),$$

where $\Phi(U_n, G, z)$ denotes the family of strategies $\psi \in \Phi$ such that $EU_n(V_T^{z,\psi} - G)$ exists.

If G is interpreted as the payoff at time T of some derivative security, the quantity $u_n(G, z)$ represents the supremum of expected utility from initial capital z when delivering G at the terminal date.

Theorem 2.11 *Suppose that S is bounded and Assumptions 2.2, 2.3, 2.4 and 2.6 hold. Then there exist almost surely unique optimal strategies $\psi^*_{n,t}(G, z),\ 1 \leq t \leq T,\ n \in \bar{\mathbb{N}}$ satisfying*

$$u_n(G, z) = EU_n(V_T^{z,\psi^*_n(G,z)} - G).$$

For all $1 \leq t \leq T$ almost surely

$$\lim_{n\to\infty} \psi^*_{n,t}(G, z) = \psi^*_{\infty,t}(G, z).$$

Moreover, $\lim_{n\to\infty} u_n(G, z) = u_\infty(G, z)$ uniformly on compact sets.

Theorem 2.12 *Assume the hypotheses of the previous Theorem, with Assumption 2.4 replaced by 2.8. For all $N > 0$ there exist suitable constants $J_t(N)$ and $J(N)$ such that, for all $1 \leq t \leq T$,*

$$\begin{aligned} \sup_{z\in[-N,N]} |\psi^*_{n,t}(G, z) - \psi^*_{\infty,t}(G, z)| &\leq J_t(N)g(n), \\ \sup_{z\in[-N,N]} |u_n(G, z) - u_\infty(G, z)| &\leq J(N)g(n). \end{aligned}$$

Remark 2.13 Without Assumption 2.3 proofs get messy and we obtain that the suitably defined projections of the optimal strategies on D_t converge to the projection of the limiting strategy.

Remark 2.14 Consider random utility functions $\mathcal{U}_n(x,\omega)$. In this paper we study the economic meaningful case where $\mathcal{U}_n(x,\omega) = U_n(x - G(\omega))$. Nevertheless results of Theorem 2.11 (resp. 2.12) can be extended to general random utility functions if we assume an almost sure analog of Assumption 2.4 (resp. 2.8) and the additional hypothesis :

$$\forall x, \quad \text{ess.} \sup_{\Omega, n\in\bar{\mathbb{N}}} |\mathcal{U}_n(x,\omega)| < \infty \text{ and ess.} \inf_{\Omega, n\in\bar{\mathbb{N}}} |\mathcal{U}_n'(0,\omega)| > 0.$$

2.4 Applications to convergence of utility based prices

Take again $G \in L_+^\infty$, interpreted as the payoff at time T of some derivative security.

A remarkable pricing method has been suggested in Davis (1997) : to evaluate claim G using the measure

$$\frac{dQ(z)}{dP} = \frac{U'(V_T^{z,\psi^*(0,z)})}{EU'(V_T^{z,\psi^*(0,z)})},$$

where U is a suitable utility function and $\psi^*(0,z)$ is the optimal strategy with initial endowment z and without delivering any claim, i.e.

$$u(0,z) = \sup_{\psi\in\Phi(U,0,z)} EU(V_T^{z,\psi}) = EU(V_T^{z,\psi^*(0,z)}).$$

Under appropriate conditions (see section 6 of Rásonyi and Stettner (2005)), $Q(z)$ indeed defines an equivalent risk-neutral measure and the Davis price defined by

$$q(G,z) = E^{Q(z)}(G)$$

is an arbitrage free price. In this way individual preferences of agents are taken into account when choosing the pricing functional by some "marginal rate of substitution argument", see Davis (1997) or p. 229 of Bingham and Kiesel (1998) for more economic justifications about this pricing rule.

Theorem 2.11 permits us to establish the continuity of Davis price with respect to changes in the agents' preferences.

Theorem 2.15 *Under the hypotheses of Theorem 2.11, the Radon-Nykodim derivatives*

$$\frac{dQ_n(z)}{dP} = \frac{U_n'(V_T^{z,\psi_n^*(0,z)})}{EU_n'(V_T^{z,\psi_n^*(0,z)})},$$

define equivalent martingale measures for S and $Q_n(z) \to Q_\infty(z)$ in the total variation norm. Consequently,

$$\lim_{n\to\infty} q_n(G,z) = q_\infty(G,z), \tag{4}$$

for any contingent claim $G \in L_+^\infty$.

Moreover, under the additional assumption of Theorem 2.12, for all $N > 0$ there exists some constant $A(N)$ such that

$$\sup_{z\in[-N,N]} |q_n(G,z) - q_\infty(G,z)| \leq A(N)g(n).$$

Now consider another pricing concept, originating from Hodges and Neuberger (1989). The Hodges-Neuberger or utility indifference price of some bounded contingent claim G is the minimal amount of money to be paid to the seller and added to her initial capital so that her utility when selling G is greater than the one she could get without selling it.

Definition 2.16 For any $G \in L_+^\infty$ and $x \in \mathbb{R}$, the utility indifference price $p_n(G,x)$ is defined as

$$p_n(G,x) = \inf\{z \in \mathbb{R} : u_n(G, x+z) \geq u_n(0,x)\},\ n \in \bar{\mathbb{N}}.$$

It is easy to check that this quantity is well-defined and $0 \leq p_n(G,x) \leq \|G\|_\infty$. In our case $u_n(G,\cdot)$, $u_n(0,\cdot)$ are strictly increasing (see the statement of Proposition 4.7), so $u_n(G, x + p_n(G,x)) = u_n(0,x)$.

Theorem 2.17 *Under the hypotheses of Theorem 2.11,*

$$\lim_{n\to\infty} p_n(G,x) = p_\infty(G,x).$$

3 Counter-examples

In this section we demonstrate the pathologies which might arise in the absence of our assumptions. In all the examples we will suppose $G = 0$ for simplicity, so our value function will be

$$u_n(x) := \sup_{\psi\in\Phi(U,0,x)} EU_n(V_T^\psi).$$

Firstly, the convergence of optimal strategies may fail for unbounded price processes, even though all the other assumptions hold.

Example 3.1 Define for all $n \in \bar{\mathbb{N}}$

$$U_n(x) := 1 - (1-x)^{2+1/n},\ x \leq 0, \quad U_n(x) := (4 + 2/n)\sqrt{x+1} - 4 - 2/n,\ x > 0,$$

with the convention $1/\infty = 0$. It is easily verified that Assumption 2.4 and 2.6 holds for this sequence. Now set

$$\alpha_1 := \sum_{k=2}^\infty \frac{1}{k^3 \log^2 k}, \quad \alpha_2 := \sum_{k=1}^\infty \frac{1}{k^2}.$$

Take $T = 1$ and ΔS_1 such that

$$P(\Delta S_1 = -k) = \frac{1}{2\alpha_1 k^3 \log^2 k},\ k \geq 2; \quad P(\Delta S_1 = \delta k) = \frac{1}{2\alpha_2 k^2},\ k \geq 1,$$

where $\delta > 0$ is to be chosen later. Assumption 2.2 holds with e.g. $\beta = 1$ and $\kappa = 1/3$. As $\sum_{k\geq 0} \frac{1}{k^\alpha \log^2 k} = \infty$ for $\alpha < 1$ and $< \infty$ for $\alpha \geq 1$, it is easy to check that for all $n \in \mathbb{N}$ and $\psi \neq 0$ we have $EU_n(\psi \Delta S_1) = -\infty$. Consequently $\psi_n^* = 0$ is optimal. On the other hand,

$$EU_\infty(\Delta S_1) = \frac{1}{2} - \sum_{k=2}^\infty \frac{(k+1)^2}{2\alpha_1 k^3 \log^2 k} + 2\sum_{k=1}^\infty \frac{\sqrt{\delta k + 1}}{\alpha_2 k^2} - 2,$$

which is finite and, for δ sufficiently large, strictly greater than 0, so ψ_∞^* (which exists by Theorem 2.7 of Rásonyi and Stettner (2005)) cannot be 0.

The following construction shows that if S fails to be bounded, the value functions u_n may converge to ∞ instead of u_∞.

Example 3.2 Let $S_0 := 0$ and

$$P(\Delta S_1 = k^4 - 1) = \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}, \quad k \geq 4, \quad \text{and} \quad P(\Delta S_1 = -1) = 1/2.$$

Define also

$$\begin{aligned} U_n(x) &= x + \frac{1}{n}, \quad x < 0, \\ U_n(x) &= \frac{1}{n}\sqrt{x+1}, \quad 0 \leq x \leq n^4 - 1, \\ U_n(x) &= n, \quad x > n^4 - 1. \end{aligned}$$

This sequence converges pointwise to

$$U_\infty(x) = x, \quad x < 0, \quad U_\infty(x) = 0, \quad x \geq 0.$$

For $x \geq 0$,

$$\begin{aligned} u_\infty(x) &= \frac{x-1}{2} 1_{x<1} \\ u_n(x) &\geq u_n(0) \geq EU_n(\Delta S_1) \geq -\frac{1}{2} + \frac{1}{2n} + nP(\Delta S_1 \geq n^4 - 1) = -\frac{1}{2} + \frac{1}{2n} + \sqrt{n}, \end{aligned}$$

showing that $u_n(x) \to \infty > u_\infty(x)$. These U_n satisfy Assumption 2.6. With some extra work it would be possible to construct a similar example with U_n satisfying Assumption 2.4, too (i.e. U_n strictly concave and smooth).

Now we point out what may go wrong in utility maximization if we drop Assumption 2.2: the value function $u(x) := \sup_{\psi \in \Phi(U,0,x)} EU(V_T^\psi)$ may be infinite even if S is bounded!

Example 3.3 Suppose that $T = 2$, $\mathcal{F}_1 = \mathcal{P}(\mathbb{N})$ and $P(\{n\}) = 1/2^n,\ n \geq 1$. Assume that $S_0 = S_1 = 0$ and

$$P(\Delta S_2 = -1/2^n | \mathcal{F}_1)(n) = 1/2 = P(\Delta S_2 = 1 | \mathcal{F}_1)(n).$$

Define also

$$U(x) = \frac{1}{2}x + 1,\ x < 0, \quad U(x) = \sqrt{x+1},\ x \geq 0.$$

Taking $\psi(n) := 2^{2n} - 1$ we clearly have

$$\begin{aligned} u(0) &\geq EU(\psi \Delta S_2) = EE(U(\psi \Delta S_2)|\mathcal{F}_1) = \sum_{n=1}^{\infty} \frac{1}{2^n}[\frac{1}{2}U(-\psi(n)/2^n) + \frac{1}{2}U(\psi(n))] \\ &= \sum_{n=1}^{\infty} \frac{1}{2^n}[2^{n-2} + \frac{1}{2} + \frac{1}{2^{n+2}}] = \infty. \end{aligned}$$

In this example one can take $\kappa_1 = 1/2$ constant, but β_1 cannot be chosen constant, hence Assumption 2.2 fails. A similar construction can be given where β_1 is constant and κ_1 is not.

4 Facts about utility maximization

4.1 Bounds on the optimal strategies

We work on the primal problem and use a dynamic programming procedure to prove the existence of optimal strategies and to derive bounds on them. If we used the dual approach (see Kramkov and Schachermayer (1999) and Schachermayer (2000)), we should find bounds on the solution of the dual problem which is even more difficult to control.

Theorem 4.4 holds true under weaker hypotheses on $(U_n)_{n\in\bar{\mathbb{N}}}$ than Assumption 2.4. What we need is the following:

Assumption 4.1 The function $U_n : \mathbb{R} \to \mathbb{R}$, $n \in \bar{\mathbb{N}}$ are concave, nondecreasing and continuously differentiable,

$$\sup_{n\in\bar{\mathbb{N}}} |U_n(x)| < \infty \text{ for all } x \in \mathbb{R}, \quad \inf_{n\in\bar{\mathbb{N}}} U_n'(0) > 0.$$

In what follows, it is crucial that the asymptotic elasticity Assumption 2.6 admits a reformulation which is preserved during the dynamic programming procedure. This is the content of the next Condition. Let $V : \mathbb{R} \to \mathbb{R}$ be a function.

Condition 4.2 There exist $C_1, C_2 > 0$ such that

$$\begin{aligned} V(\lambda x) &\leq \lambda^\gamma V(x + C_1) + C_2\lambda^\gamma, \\ V(\lambda x) &\leq \lambda V(x + C_1) + C_2\lambda^\gamma, \end{aligned}$$

for all $x \in \mathbb{R}$ and $\lambda \geq 1$.

Fix some $G \in L_+^\infty$ and set $U_{n,T}(x,\omega) := U_n(x - G(\omega))$. Proposition 4.3 below initiates the dynamic programming.

Proposition 4.3 *Under Assumptions 2.6 and 4.1, $U_{n,T}$ satisfies Condition 4.2 almost surely with constants C_1, C_2 independent from n.*

Proof. Set $C_1 := \|G\|_\infty$, $C_3(x) := \sup_{n\in\bar{\mathbb{N}}} |U_n(x)|$ and $C_4 := C_3(0)$. Define $\tilde{U}_n(x) := U_n(x) - U_n(0)$. Then by Assumption 2.6 we have for $x \geq \tilde{x}$ and $\lambda \geq 1$:

$$\tilde{U}_n(\lambda x) \leq \lambda^\gamma U_n(x) + C_4 \leq \lambda^\gamma \tilde{U}_n(x) + C_4\lambda^\gamma + C_4 \leq \lambda^\gamma \tilde{U}_n(x) + 2C_4\lambda^\gamma.$$

For $0 \leq x \leq \tilde{x}$, using monotonicity:

$$\begin{aligned} \tilde{U}_n(\lambda x) &\leq \tilde{U}_n(\lambda\tilde{x}) \leq \lambda^\gamma U_n(\tilde{x}) + C_4 \leq \lambda^\gamma C_3(\tilde{x}) + C_4 \\ &\leq \lambda^\gamma \tilde{U}_n(x) + \lambda^\gamma[C_4 + C_3(\tilde{x})], \end{aligned}$$

since $\tilde{U}_n(x) \geq 0$ if $x \geq 0$. For $x \leq 0$, by concavity:

$$\begin{aligned} \tilde{U}_n(\lambda x) &\leq \tilde{U}_n(x) + \tilde{U}_n'(x)(\lambda - 1)x \leq \tilde{U}_n(x) + \frac{\tilde{U}_n(x) - \tilde{U}_n(0)}{x}(\lambda - 1)x \\ &\leq \lambda\tilde{U}_n(x) \leq \lambda^\gamma \tilde{U}_n(x). \end{aligned}$$

Putting together the estimations so far, we obtain that Condition 4.2 holds for $\tilde{U}_n$, $n \in \bar{\mathbb{N}}$ with uniform constants $\tilde{C}_1 := 0$, $\tilde{C}_2 := 2C_4 + C_3(\tilde{x})$. Now for $U_{n,T}$ we get

$$\begin{aligned} U_{n,T}(\lambda x) &\leq U_n(\lambda x) \leq \tilde{U}_n(\lambda x) + C_4 \leq \lambda^\gamma \tilde{U}_n(x) + [\tilde{C}_2 + C_4]\lambda^\gamma \\ &\leq \lambda^\gamma U_n(x) + [\tilde{C}_2 + 2C_4]\lambda^\gamma \leq \lambda^\gamma U_{n,T}(x + C_1) + [\tilde{C}_2 + 2C_4]\lambda^\gamma, \end{aligned}$$

showing that the first inequality of Condition 4.2 is true for $U_{n,T}$, $n \in \bar{\mathbb{N}}$ with the choice $C_2 := \tilde{C}_2 + 2C_4$, uniformly in n. The second inequality follows in the same way. □

Theorem 4.4 *Suppose that Assumptions 2.2, 2.6 and 4.1 hold. For all $n \in \bar{\mathbb{N}}$, we introduce the following random functions :*

$$\begin{aligned} U_{n,T}(x) &:= U_n(x-G), \\ U_{n,s}(x) &:= \operatorname{ess.}\sup_{\xi \in \Xi_s} E(U_{n,s+1}(x+\langle \xi, \Delta S_{s+1}\rangle)|\mathcal{F}_s), \quad 0 \le s \le T-1. \end{aligned}$$

For all $n \in \bar{\mathbb{N}}$, $0 \le s \le T$, $U_{n,s}$ are well-defined and satisfy

$$U_{n,s}(x) < \infty. \tag{5}$$

The functions $U_{n,s}$ have almost surely concave and increasing continuously differentiable versions satisfying Condition 4.2 with constants uniform in n.

For all $n \in \bar{\mathbb{N}}$, $0 \le s \le T-1$ and $x \in \mathbb{R}$, there exists $\tilde{\xi}_{n,s+1}(x) \in \Xi_s$ such that $\tilde{\xi}_{n,s+1} \in D_{s+1}$ a.s. and

$$U_{n,s}(x) = E(U_{n,s+1}(x + \langle \tilde{\xi}_{n,s+1}(x), \Delta S_{s+1}\rangle)|\mathcal{F}_s). \tag{6}$$

For all $0 \le s \le T-1$, there exist nondecreasing functions M_s, $\hat{M}_s$ and $H_s : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $n \in \bar{\mathbb{N}}$, $x \in \mathbb{R}$:

$$\begin{aligned} |\tilde{\xi}_{n,s+1}(x)| &\le \hat{M}_{s+1}(|x|), &(7)\\ U_n(x - M_{s+1}(|x|)) &\le U_{n,s+1}(x) \le U_n(x + M_{s+1}(|x|)), &(8)\\ U'_{n,s}(x) &= E(U'_{n,s+1}(x + \langle \tilde{\xi}_{n,s+1}(x), \Delta S_{s+1}\rangle)|\mathcal{F}_s), &(9)\\ U'_n(H_{s+1}(|x|)) &\le U'_{n,s+1}(x) \le U'_n(-H_{s+1}(|x|)). &(10) \end{aligned}$$

For all $n \in \bar{\mathbb{N}}$, $z \in \mathbb{R}$ the utility maximization problems

$$EU_n(V_T^{z,\psi} - G) \to \max., \ \psi \in \Phi(U_n, G, z),$$

admit optimal solutions $\psi_n^*(z)$ given by

$$\psi^*_{n,1}(z) := \tilde{\xi}_{n,1}(z), \quad \psi^*_{n,t+1}(z) := \tilde{\xi}_{n,t+1}(z + \sum_{k=1}^{t} \langle \psi^*_{n,k}(z), \Delta S_k \rangle). \tag{11}$$

There exists nondecreasing functions $\Upsilon_t : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $n \in \bar{\mathbb{N}}$, $z \in \mathbb{R}$

$$|\psi^*_{n,t}(z)| \le \Upsilon_t(|z|). \tag{12}$$

and the value functions of the optimization problems are finite, i.e.

$$u_n(G,z) = U_{n,0}(z) < \infty.$$

Remark 4.5 For the sake of notational simplicity we do not index the optimal solutions by G, *i.e.* we denoted $\psi^*_{n,t}(G,z)$ by $\psi^*_{n,t}(z)$ and $\tilde{\xi}_{n,t}(G,x)$ by $\tilde{\xi}_{n,t}(x)$, for all $1 \le t \le T$ and $n \in \bar{\mathbb{N}}$.

Proof. Suppose $d = 1$ for simplicity and let R denote a constant bound for the process $|\Delta S|$.

Sections 4 and 5 of Rásonyi and Stettner (2005) will be used, but the estimations have to be carried out in a more explicit way. The hypotheses which are needed there:

$E(U_{n,s}(x)) > -\infty$, (5) and Condition 4.2 for $U_{n,s}$; these will be shown in the reasonings below.

First note that $E(U_{n,s}(x)) > -\infty$ holds true by backward induction for all $s = 0, \dots, T$ because $U_{n,s}(x) \geq E(U_{n,s+1}(x)|\mathcal{F}_s)$ and $U_{n,T}(x) \geq U_n(x - \|G\|_\infty)$. From standard arguments, $U_{n,s}$ are concave functions.

We shall apply backward induction to prove the statements (5) to (10). First, (5) is trivial for $s = T$, (8) and (10) are trivial for $s = T - 1$ and Condition 4.2 for $U_{n,T}$ holds by Proposition 4.3. Moreover, as S and G are bounded, it is easy to see that (5) holds true for $s = T - 1$. So from Proposition 4.4, Lemma 4.9 and Proposition 6.5 of Rásonyi and Stettner (2005), $U_{n,T-1}$ have almost surely concave, increasing, continuously differentiable versions and (6), (9) hold. Finally, (7) will follow just like in the induction step below.

Let us proceed supposing that the induction hypotheses hold for $s \geq t$. We get from (7) for $s = t$ that

$$x + \tilde{\xi}_{n,t+1}(x)\Delta S_{t+1} \in [x - \hat{M}_{t+1}(|x|)R, x + \hat{M}_{t+1}(|x|)R],$$

and from (8) for $s = t$

$$U_{n,t+1}(x + \hat{M}_{t+1}(|x|)R) \leq U_n \left(x + \hat{M}_{t+1}(|x|)R + M_{t+1}(|x| + \hat{M}_{t+1}(|x|)R) \right)$$

because M_{t+1} and U_n are nondecreasing. Also

$$U_{n,t+1}(x - \hat{M}_{t+1}(|x|)R) \geq U_n \left(x - \hat{M}_{t+1}(|x|)R - M_{t+1}(|x| + \hat{M}_{t+1}(|x|)R) \right).$$

Defining

$$M_t(u) := \hat{M}_{t+1}(u)R + M_{t+1}(u + \hat{M}_{t+1}(u)R),\ u \in \mathbb{R}_+,$$

M_t is nondecreasing as $\hat{M}_{t+1}$ and M_{t+1} are. Using (6) for $s = t$ and the fact that $U_{n,t+1}$ is nondecreasing, we get that almost surely

$$U_n(x - M_t(|x|)) \leq U_{n,t}(x) \leq U_n(x + M_t(|x|)), \tag{13}$$

showing (8) for $s = t-1$. Moreover, as S is bounded, it is easy to see that (5) holds true for $s = t - 1$. Then Condition 4.2 holds for $U_{n,t-1}$ with the same constants as in Proposition 4.3, by the argument of Proposition 5.2 of Rásonyi and Stettner (2005). So we can again apply Proposition 4.4 and Lemma 4.9 of the cited article and (6) holds for $s = t - 1$ and $U_{n,t-1}$ have almost surely concave and nondecreasing versions. Moreover, we get that from Proposition 6.5 of the same paper that $U_{n,t-1}$ has almost surely continuously differentiable versions and (9) is satisfied.

It is also clear from (7), (9), (10) for $s = t$ and from the facts that H_{t+1} is nondecreasing and $U'_{n,t+1}$ nonincreasing:

$$U'_{n,t}(x) = E(U'_{n,t+1}(x + \tilde{\xi}_{n,t+1}(x)\Delta S_{t+1})|\mathcal{F}_t) \geq U'_n(H_{t+1}(|x| + R\hat{M}_{t+1}(|x|))),$$

This, together with an upper estimate of the same kind, shows (10) for $s = t - 1$ with the choice

$$H_t(u) := H_{t+1}(u + R\hat{M}_{t+1}(u)),\ u \in \mathbb{R}_+.$$

Now we want to prove that a bounded optimal strategy $\tilde{\xi}_{n,t}(x)$ exists. Let $y > 0$. As $U_{n,t}$ is concave,

$$U_{n,t}(-y) \quad \leq \quad U_{n,t}(0) - yU'_{n,t}(0). \tag{14}$$

Using condition (8) for $s = t-1$ we see that $U_{n,t}(0) \leq U_n(M_t(0))$, and from Assumption 4.1 we get that

$$\sup_{n\in\bar{\mathbb{N}}} U_{n,t}(0) < \infty. \tag{15}$$

We now prove that $\inf_{n\in\bar{\mathbb{N}}} U'_{n,t}(0) > 0$. For this purpose, introduce the following sets:

$$A_{n,s+1} = \{\tilde{\xi}_{n,s+1}(0)\Delta S_{s+1} \leq 0\},\ s \geq t.$$

From Assumption 2.2, $P(A_{n,s+1}|\mathcal{F}_s) \geq \kappa$. Apply (9) for $s \geq t$:

$$\begin{aligned} U'_{n,t}(0) &= E(U'_{n,t+1}(\tilde{\xi}_{n,t+1}(0)\Delta S_{t+1})|\mathcal{F}_t) \geq E(I_{A_{n,t+1}} U'_{n,t+1}(0)|\mathcal{F}_t) \\ &\geq E(I_{A_{n,t+1}} \dots I_{A_{n,T}} U'_{n,T}(0)|\mathcal{F}_t), \end{aligned}$$

iterating the same reasoning. We obtain that

$$U'_{n,t}(0) \;\; \geq \;\; U'_n(0) E(I_{A_{n,t+1}} \dots I_{A_{n,T}}|\mathcal{F}_t) \geq \kappa^{T-t} \inf_{n\in\bar{\mathbb{N}}} U'_n(0),$$

which is strictly positive by Assumption 4.1. So by (14) and (15) there exists a constant N_t (independent from n) such that $U_{n,t}(-N_t) < -1$ with probability one, for all $n \in \bar{\mathbb{N}}$.

Apply the estimations of the proof of Lemma 4.8 in Rásonyi and Stettner (2005) with the choice $V := U_t$ to an arbitrary $\xi \in \Xi_{t-1},\ \xi \in D_t,\ |\xi| \neq 0$. In that Lemma C_1 is taken to be 0, but the argument can be easily adapted to yield

$$E(U_{n,t}(x+\xi\Delta S_t)|\mathcal{F}_{t-1}) \leq |\xi|^\gamma L_{n,t}(x) + 2C_2|\xi|^\gamma - |\xi|^{(1+\gamma)/2}\kappa/2, \tag{16}$$

whenever

$$C_1 + \frac{|x|}{|\xi|^{(1+\gamma)/2}} - |\xi|^{(1-\gamma)/2}\beta < -N_t,$$

here $L_{n,t}(x)$ is a random variable such that

$$\begin{aligned} 0 \leq L_{n,t}(x) &\leq 2U^+_{n,t}(x + C_1 + R) \leq 2U^+_n(|x| + C_1 + R + M_t(|x| + C_1 + R)) \\ &\leq 2\sup_{n\in\bar{\mathbb{N}}} U^+_n(|x| + C_1 + R + M_t(|x| + C_1 + R)) =: G_t(|x|), \end{aligned}$$

and the latter is a deterministic function, nondecreasing in $|x|$ and independent of n, by Assumption 4.1.

Now there exists some deterministic function $u \to \hat{M}_t(u)$, $u \in \mathbb{R}_+$ (chosen to be nondecreasing) such that if $|\xi(\omega)| > \hat{M}_t(|x|)$ then

$$\begin{aligned} |\xi(\omega)|^\gamma G_t(|x|) + 2C_2|\xi(\omega)|^\gamma - |\xi(\omega)|^{(1+\gamma)/2}\kappa/2 &< \inf_{n\in\bar{\mathbb{N}}} U_n(x - M_t(|x|)), \\ C_1 + \frac{|x|}{|\xi(\omega)|^{(1+\gamma)/2}} - |\xi(\omega)|^{(1-\gamma)/2}\beta &< -N_t, \end{aligned}$$

here the infimum is finite by Assumption 4.1 again. Define the set $A = \{|\xi| > \hat{M}_t(|x|)\} \in \mathcal{F}_{t-1}$. From (16) and (8) for $s = t-1$ we have that on A,

$$E(U_{n,t}(x+\xi\Delta S_t)|\mathcal{F}_{t-1}) < U_n(x - M_t(|x|)) \leq E(U_{n,t}(x)|\mathcal{F}_{t-1}),$$

hence

$$\begin{aligned} E(U_{n,t}(x+\xi\Delta S_t)|\mathcal{F}_{t-1}) &\leq I_A E(U_{n,t}(x)|\mathcal{F}_{t-1}) + I_{A^c} E(U_{n,t}(x+\xi\Delta S_t)|\mathcal{F}_{t-1}) \\ &\leq E(U_{n,t}(x+\xi I_{A^c}\Delta S_t)|\mathcal{F}_{t-1}), \end{aligned}$$

with strict inequality on A. Assume that $P(A) > 0$ and apply the last inequality for $\xi = \tilde{\xi}_{n,t}(x)$; then the strategy $\tilde{\xi}_{n,t}(x)I_{A^c}$ contradicts optimality. So (7) holds for $s = t-1$.

It remains to prove that the strategies defined by (11) are optimal. Just like in Proposition 5.3 of Rásonyi and Stettner (2005), we obtain that for any trading strategy $\psi \in \Phi(U_n, G, z)$:

$$E(U_n(V_T^{z,\psi})|\mathcal{F}_0) \leq U_{n,0}(z) = E(U_n(V_T^{z,\psi_n^*(z)})|\mathcal{F}_0).$$

As $U_{n,0}(z)$ is finite and $\mathcal{F}_0$ is trivial one gets that $u_n(G,z) = U_{n,0}(z) < \infty$, and for all $\psi \in \Phi(U_n, G, z)$, $E(U_n(V_T^{z,\psi})) \leq E(U_n(V_T^{z,\psi_n^*(z)})) = u_n(G,z) < \infty$. Thus $\psi_n^*(z)$ is the solution of

$$EU_n(V_T^{z,\psi}) \to \max., \ \psi \in \Phi(U_n, G, z).$$

By induction, it is easy to see from (7) that (12) holds with

$$\Upsilon_1(u) = \hat{M}_1(u) \text{ and } \Upsilon_{t+1}(u) = \hat{M}_{t+1}\left(u + R\sum_{s=1}^{t}\Upsilon_s(u)\right).$$

□

Corollary 4.6 *Under the conditions of Theorem 4.4, there exist nondecreasing functions $F_t : \mathbb{R}_+ \to \mathbb{R}_+,\ 0 \leq t \leq T$ such that for all $n \in \bar{\mathbb{N}}$*

$$|V_t^{z,\psi_n^*(z)}| \leq F_t(|z|) \text{ a.s.}$$

for the optimal strategies $\psi_n^*(z)$ constructed in the previous Theorem.

Proof. Indeed, define $F_t(u) := u + R\left[\sum_{j=1}^t \Upsilon_j(u)\right]$. □

4.2 Uniqueness

Proposition 4.7 *If we assume, in addition to conditions of Theorem 4.4, that the U_n are strictly concave for $n \in \bar{\mathbb{N}}$ then the $U_{n,t}$ (and thus $u_n(G,\cdot) = U_{n,0}$) are strictly concave a.s. for all $t = 0,\dots,T$ and there exists a unique optimal strategy ψ_n^* such that almost surely $\psi_{n,t}^* \in D_t$, for all $t = 1,\dots,T$.*

Proof. To see strict concavity we argue by backward induction : the case $s = T$ is trivial, suppose that for some $s < T$, $x \neq y$ and $0 < \alpha < 1$ we have

$$U_{n,s}(\alpha x + (1-\alpha)y) = \alpha U_{n,s}(x) + (1-\alpha)U_{n,s}(y),$$

on a set $A \in \mathcal{F}_s$ of positive probability. By concavity of $U_{n,s+1}$ and optimality of $\tilde{\xi}_{n,s+1}(\alpha x + (1-\alpha)y)$ we have

$$\begin{aligned} E(\alpha U_{n,s+1}(x+\tilde{\xi}_{n,s+1}(x)\Delta S_{s+1}) + (1-\alpha)U_{n,s+1}(y+\tilde{\xi}_{n,s+1}(y)\Delta S_{s+1})|\mathcal{F}_s) &\leq \\ E(U_{n,s+1}(\alpha x + (1-\alpha)y + [\alpha\tilde{\xi}_{n,s+1}(x) + (1-\alpha)\tilde{\xi}_{n,s+1}(y)]\Delta S_{s+1})|\mathcal{F}_s) &\leq \\ E(U_{n,s+1}(\alpha x + (1-\alpha)y + \tilde{\xi}_{n,s+1}(\alpha x + (1-\alpha)y)\Delta S_{s+1})|\mathcal{F}_s). & \end{aligned}$$

On A, the first and the third lines are equal, so from the equality of the first and the second lines we get

$$I_A\left(\alpha U_{n,s+1}(x+\tilde{\xi}_{n,s+1}(x)\Delta S_{s+1})+(1-\alpha)U_{n,s+1}(y+\tilde{\xi}_{n,s+1}(y)\Delta S_{s+1})\right)= \\ I_A U_{n,s+1}(\alpha x+(1-\alpha)y+[\alpha\tilde{\xi}_{n,s+1}(x)+(1-\alpha)\tilde{\xi}_{n,s+1}(y)]\Delta S_{s+1}).$$

On A one has, by strict concavity of $U_{n,s+1}$,

$$x+\tilde{\xi}_{n,s+1}(x)\Delta S_{s+1}=y+\tilde{\xi}_{n,s+1}(y)\Delta S_{s+1}.$$

As $x\neq y$, the quantity $\tilde{\xi}_{n,s+1}(x)-\tilde{\xi}_{n,s+1}(y)$ is nonzero, so we get on A,

$$\frac{x-y}{\tilde{\xi}_{n,s+1}(x)-\tilde{\xi}_{n,s+1}(y)}=\Delta S_{s+1}.$$

Thus $D_{s+1}\neq\{0\}$ on A. Moreover the left-hand side is $\mathcal{F}_s$-measurable, so we arrive at a contradiction as ΔS_{s+1} has nondegenerate conditional distribution by Assumption 2.2. Unicity of $\tilde{\xi}_{n,t}$ is a consequence of Theorem 2.8 in Rásonyi and Stettner (2005). □

5 Facts about convergence

Corollary 5.1 *Suppose that Assumptions 2.2, 2.4 and 2.6 hold. Then $U_{n,t}$ converges to $U_{\infty,t}$ almost surely, uniformly on compacts, for all $0\leq t\leq T$.*
In particular, $u_n(G,\cdot)=U_{n,0}(\cdot)$ converges to $u_\infty(G,\cdot)=U_{\infty,0}(\cdot)$ uniformly on compacts.

Proof. It suffices to establish almost sure convergence pointwise as by monotonicity and concavity of $U_{n,t}$ this entails almost sure uniform convergence on compact sets, see p. 90 of Rockafellar (1970). Assumption 2.4 and strict monotonicity of U_∞ imply that Assumption 4.1 holds and hence Theorem 4.4 applies. It is clear from (6) that

$$U_{n,t}(x)=E(U_n(x+\sum_{i=t+1}^{T}\langle\phi^*_{n,i},\Delta S_i\rangle)|\mathcal{F}_t),$$

where

$$\phi^*_{n,t+1}:=\tilde{\xi}_{n,t+1}(x),\quad \phi^*_{n,j}:=\tilde{\xi}_{n,j}(x+\sum_{i=t+1}^{j-1}\langle\phi^*_{n,i},\Delta S_i\rangle),\ j>t+1.$$

Define

$$l_n:=x+\sum_{i=t+1}^{T}\langle\phi^*_{n,i},\Delta S_i\rangle,\ n\in\bar{\mathbb{N}}.$$

Then we have

$$\begin{aligned}\liminf_{n\to\infty}U_{n,t}(x) &= \liminf_{n\to\infty}E(U_n(l_n)|\mathcal{F}_t)\\ &\geq \liminf_{n\to\infty}E(U_n(l_\infty)|\mathcal{F}_t)=E(U_\infty(l_\infty)|\mathcal{F}_t)=U_{\infty,t}(x),\end{aligned}$$

by optimality of ϕ^*_n, Assumption 2.4, Remark 2.5 and the fact that the random variable l_∞ is bounded by (12).

In fact, all the l_n are bounded, uniformly in $n \in \bar{\mathbb{N}}$ (we will denote by K such a bound) and recalling (8), the random variables $U_{n,t}(x) = E(U_n(l_n)|\mathcal{F}_t)$ are also bounded, uniformly in $n \in \bar{\mathbb{N}}$. Hence by Lemma 2 of Kabanov and Stricker (2001), there exists an $\mathcal{F}_t$-measurable random subsequence σ_n such that

$$\limsup_{n\to\infty} U_{n,t}(x) = \lim_{n\to\infty} U_{\sigma_n,t}(x).$$

Using again Lemma 2 of [5] for the uniformly bounded sequence l_{σ_n} we can extract another random subsequence (for which we will keep the same notation) such that l_{σ_n} converges to some l^*.

$$\begin{aligned} |E(U_{\sigma_n}(l_{\sigma_n})|\mathcal{F}_t) - E(U_\infty(l^*)|\mathcal{F}_t)| &\leq |E(U_{\sigma_n}(l_{\sigma_n})|\mathcal{F}_t) - E(U_\infty(l_{\sigma_n})|\mathcal{F}_t)| + \\ &\quad |E(U_\infty(l_{\sigma_n})|\mathcal{F}_t) - E(U_\infty(l^*)|\mathcal{F}_t)|. \end{aligned}$$

The first term is $o(1)$ using the uniform convergence on compact sets of U_n to U_∞ and the fact that l_{σ_n} are uniformly bounded by K. As $l_{\sigma_n} \to l^*$, U_∞ is continuous, $|U_\infty(l_{\sigma_n})|$ is uniformly bounded, we can use Lebesgue's theorem and the second term is also $o(1)$. As the set of portfolio values is closed in probability (see e.g. the argument of Theorem 1 in Kabanov and Stricker (2001)), l^* is itself the value of a portfolio. Now

$$\limsup_{n\to\infty} U_{n,t}(x) = \lim_{n\to\infty} EU_{\sigma_n}(l_{\sigma_n}) = E(U_\infty(l^*)|\mathcal{F}_t) \leq E(U_\infty(l_\infty)|\mathcal{F}_t) = U_{\infty,t}(x),$$

by optimality of l_∞, finishing the proof of this Corollary. □

The following Lemma will be used to establish the rate of convergence of the optimal strategies.

Lemma 5.2 *Suppose that S is bounded, Assumptions 2.2, 2.3, 2.6 and 2.8 hold. Consider $\tilde{\xi}_{n,s}(x)$, $n \in \bar{\mathbb{N}}$, $1 \leq s \leq T$ as defined in Theorem 4.4. Then for all $N > 0$, almost surely,*

$$\sup_{x\in[-N,N]} |U'_{n,s}(x) - U'_{\infty,s}(x)| \leq C_s(N)g(n), \ n \in \mathbb{N}, \tag{17}$$

$$\ell_s(N) \leq |U''_{n,s}(x)| \leq L_s(N), \ x \in [-N,N], \ n \in \bar{\mathbb{N}}, \tag{18}$$

$$\sup_{x\in[-N,N]} |\tilde{\xi}_{n,s}(x) - \tilde{\xi}_{\infty,s}(x)| \leq K_s(N)g(n), \ n \in \mathbb{N}, \tag{19}$$

$$\sup_{x\in[-N,N]} |U_{n,s}(x) - U_{\infty,s}(x)| \leq \tilde{C}_s(N)g(n), \ n \in \mathbb{N}, \tag{20}$$

with suitable constants $\ell_s(N)$, $L_s(N)$, $C_s(N), K_s(N), \tilde{C}_s(N) > 0$ and for all $0 \leq s \leq T$.

Proof. Assumption 2.3 assures the uniqueness of the optimal strategy by Proposition 4.7, which will crucial in the arguments of Sublemma below.

We remark that under Assumption 2.8, Assumption 4.1 is satisfied, so Theorem 4.4 applies. From now on we suppose $d = 1$ for the sake of simplicity. Let R be a constant bound for the process $|\Delta S|$. The proof is by backward induction. (20), (17) and (18) are clear for $s = T$ from Assumption 2.8 and Remark 2.9, (19) follows just like in the induction step below, so let us proceed to the induction step immediately.

Assume that (20), (17), (18) and (19) hold for $s \geq t$. Let us establish them for $s = t-1$. Let $N > 0$ and $x \in [-N,N]$, we apply (6), (9) and (7) of Theorem 4.4 for $s = t-1$ and set

$X_t = N + R\hat{M}_t(N)$. Then, using the induction hypotheses, (17) holds true because of

$$\begin{array}{rl}
|U'_{n,t-1}(x) - U'_{\infty,t-1}(x)| & \leq \\
E(|U'_{n,t}(x + \tilde{\xi}_{n,t}(x)\Delta S_t) - U'_{\infty,t}(x + \tilde{\xi}_{\infty,t}(x)\Delta S_t)||\mathcal{F}_{t-1}) & \leq \\
E(|U'_{n,t}(x + \tilde{\xi}_{n,t}(x)\Delta S_t) - U'_{\infty,t}(x + \tilde{\xi}_{n,t}(x)\Delta S_t)||\mathcal{F}_{t-1}) & \\
+E(|U'_{\infty,t}(x + \tilde{\xi}_{n,t}(x)\Delta S_t) - U'_{\infty,t}(x + \tilde{\xi}_{\infty,t}(x)\Delta S_t)||\mathcal{F}_{t-1}) & \leq \\
C_t(X_t)g(n) + E(|\Delta S_t(\tilde{\xi}_{n,t}(x) - \tilde{\xi}_{\infty,t}(x))| \sup_{y\in[-X_t,X_t]} |U''_{\infty,t}(y)||\mathcal{F}_{t-1}) & \leq \\
C_t(X_t)g(n) + L_t(X_t)RK_t(N)g(n) =: C_{t-1}(N)g(n). &
\end{array}$$

Let us define the random functions

$$f_{n,t}(x,\xi) := E(U'_{n,t}(x + \xi\Delta S_t)\Delta S_t|\mathcal{F}_{t-1}), \quad x,\xi \in \mathbb{R},\ n \in \bar{\mathbb{N}}.$$

Sublemma 5.3 *We claim that for all $n \in \bar{\mathbb{N}}$ the random functions $f_{n,t}$ have continuous differentiable versions (in both variables). $U_{n,t-1}$ have twice continuously differentiable versions (in x), $\tilde{\xi}_{n,t}(x)$ have continuously differentiable versions (in x). Furthermore,*

$$\begin{aligned}
\tilde{\xi}'_{n,t}(x) &= -\frac{\partial_1 f_{n,t}(x,\tilde{\xi}_{n,t})}{\partial_2 f_{n,t}(x,\tilde{\xi}_{n,t})}, && (21)\\
\partial_1 f_{n,t}(x,\xi) &= E(U''_{n,t}(x + \xi\Delta S_t)\Delta S_t|\mathcal{F}_{t-1}), && (22)\\
\partial_2 f_{n,t}(x,\xi) &= E(U''_{n,t}(x + \xi\Delta S_t)(\Delta S_t)^2|\mathcal{F}_{t-1}), && (23)\\
U''_{n,t-1}(x) &= E(U''_{n,t}(x + \tilde{\xi}_{n,t}(x)\Delta S_t)(1 + \tilde{\xi}'_{n,t}(x)\Delta S_t)|\mathcal{F}_{t-1}). && (24)
\end{aligned}$$

Proof of Sublemma. Continuous differentiability of $f_{n,t}$ as well as the form of the derivatives can be established in the same way as Proposition 6.4 of Rásonyi and Stettner (2005), using the bounds in Theorem 4.4 and the induction hypotheses of Lemma 5.2. Then (22) and (23) follow.

Smooth version of $\tilde{\xi}_{n,t}$ will be provided by the implicit function theorem. To see this, notice that by optimality of $\tilde{\xi}_{n,t}(x)$ and regularity of $f_{n,t}$,

$$\forall x\ f_{n,t}(x,\tilde{\xi}_{n,t}(x)) = 0, \tag{25}$$

on a set of probability one. Moreover, by strict concavity of $U_{n,t}$, $\tilde{\xi}_{n,t}(x)$ is the unique solution of equation (25). For all $N > 0$,

$$\begin{aligned}
|\partial_2 f_{n,t}(x,\xi)| &\geq \ell_t(N + R|\xi|)E((\Delta S_t)^2|\mathcal{F}_{t-1})\\
&\geq \ell_t(N + R|\xi|)E((\Delta S_t)^2 1_{\{\Delta S_t>\beta\}}|\mathcal{F}_{t-1})\\
&\geq \beta^2\ell_t(N + R|\xi|)P(\Delta S_t > \beta|\mathcal{F}_{t-1})\\
&\geq \kappa\beta^2\ell_t(N + R|\xi|) > 0,\ x \in [-N,N],
\end{aligned}$$

by (18) and Assumption 2.2. Hence by the implicit function theorem (see p. 150 of Zeidler (1986)) there exist continuously differentiable (random) functions $\zeta_n : \mathbb{R} \to \mathbb{R}$ such that on a set of probability one

$$\forall y \quad f_{n,t}(y,\zeta_n(y)) = 0.$$

Indeed, the result holds true in some neighbourhood of any real point and by unicity of the implicit function it remains true on the whole real line. Again, by unicity of the solution of equation (25) we necessarily have

$$\forall x\ \zeta_n(x) = \tilde{\xi}_{n,t}(x) \text{ a.s.}$$

so $\tilde{\xi}_{n,t}$ can be choosen to be continuously differentiable in x. Finally, $U''_{n,t-1}$ exists and is of the form (24) by (9) for $s = t-1$ and arguments akin to those of Proposition 6.4 in Rásonyi and Stettner (2005). One has to establish that Lebesgue's theorem applies when taking the derivative behind the expectation: (21), the estimates (7), (18) and Assumption 2.2 testify that

$$U''_{n,t}(x + \tilde{\xi}_{n,t}(x)\Delta S_t)(1 + \tilde{\xi}'_{n,t}(x)\Delta S_t)$$

is uniformly bounded when x stays in a compact, so we may indeed differentiate under the expectation. □

Now we turn our attention to (18) for $s = t-1$. Define the new measures W_n by

$$\begin{aligned} \alpha_n &:= -EU''_{n,t}(x + \tilde{\xi}_{n,t}(x)\Delta S_t), \\ w_n = \frac{dW_n}{dP} &:= \frac{-U''_{n,t}(x + \tilde{\xi}_{n,t}(x)\Delta S_t)}{\alpha_n}, \\ \chi_{n,t-1} &:= E(\alpha_n w_n|\mathcal{F}_{t-1}). \end{aligned}$$

First note that $\chi_{n,t-1} \geq \ell_t(N + R\hat{M}_t(N))$, by (7) and (18) for $s = t$. If we denote W-conditional expectation and variance by $E^W(\cdot|\mathcal{F}_{t-1})$ and $D^2_W(\cdot|\mathcal{F}_{t-1})$, we get

$$\begin{aligned} \frac{E^{W_n}(\Delta S_t|\mathcal{F}_{t-1})^2\chi_{n,t-1}}{E^{W_n}((\Delta S_t)^2|\mathcal{F}_{t-1})} &= \frac{E(w_n\Delta S_t|\mathcal{F}_{t-1})^2 E(\alpha_n w_n|\mathcal{F}_{t-1})E(w_n|\mathcal{F}_{t-1})}{E(w_n|\mathcal{F}_{t-1})^2 E(w_n(\Delta S_t)^2|\mathcal{F}_{t-1})} \\ &= \alpha_n \frac{E(w_n\Delta S_t|\mathcal{F}_{t-1})^2}{E(w_n(\Delta S_t)^2|\mathcal{F}_{t-1})}. \end{aligned}$$

From (24) we get that for $x \in [-N, N]$

$$\begin{aligned} -U''_{n,t-1}(x) &= E\left(\alpha_n w_n \left(1 - \frac{E(\alpha_n w_n \Delta S_t|\mathcal{F}_{t-1})}{E(\alpha_n w_n(\Delta S_t)^2|\mathcal{F}_{t-1})}\Delta S_t\right)|\mathcal{F}_{t-1}\right) \\ &= \chi_{n,t-1} - \frac{E^{W_n}(\Delta S_t|\mathcal{F}_{t-1})^2\chi_{n,t-1}}{E^{W_n}((\Delta S_t)^2|\mathcal{F}_{t-1})} \\ &= \chi_{n,t-1}\frac{D^2_{W_n}(\Delta S_t|\mathcal{F}_{t-1})}{E^{W_n}((\Delta S_t)^2|\mathcal{F}_{t-1})} \\ &\geq \ell_t(N + R\hat{M}_t(N))\frac{1}{R^2}D^2_{W_n}(\Delta S_t|\mathcal{F}_{t-1}), \end{aligned}$$

The right-hand side is greater than or equal to

$$\frac{\ell_t^2(N + R\hat{M}_t(N))}{L_t(N + R\hat{M}_t(N))}\frac{1}{R^2}\beta^2\kappa =: \ell_{t-1}(N),$$

by Assumption 2.2 and

$$w_n \geq \frac{\ell_t(N + \hat{M}_t(N)R)}{L_t(N + \hat{M}_t(N)R)},$$

which is true again by (18) for $s = t$. This shows the first inequality of (18) for $s = t-1$. The proof of the second inequality is easy and hence omitted. We know from Assumption 2.2, (23) and (18) (which has just been proved for $t-1$) that for all $n \in \bar{\mathbb{N}}$:

$$\frac{1}{\inf_{n,|\xi|\leq \hat{M}_{t-1}(N),|x|\leq N}|\partial_2 f_{n,t-1}(x,\xi)|} \leq \frac{1}{\kappa\beta^2\ell_{t-1}(N + R\hat{M}_{t-1}(N))} =: m_{t-1}.$$

By the Lagrange mean-value theorem applied to $\xi \to f_{n,t-1}(x,\xi)$, one has for $x \in [-N,N]$

$$\begin{aligned}
|\tilde{\xi}_{n,t-1}(x) - \tilde{\xi}_{\infty,t-1}(x)| &\leq m_{t-1}|f_{n,t-1}(x,\tilde{\xi}_{n,t-1}(x)) - f_{n,t-1}(x,\tilde{\xi}_{\infty,t-1}(x))| \\
&= m_{t-1}|f_{\infty,t-1}(x,\tilde{\xi}_{\infty,t-1}(x)) - f_{n,t-1}(x,\tilde{\xi}_{\infty,t-1}(x))| \\
&\leq m_{t-1}C_{t-1}(N + \hat{M}_{t-1}(N)R)Rg(n) =: K_{t-1}(N)g(n),
\end{aligned}$$

where we used (25) for the equality and (17) for $s = t-1$ in the second inequality. This ends the proof of (19) for $s = t-1$. Let $x \in [-N,N]$. Then (20) follows from

$$|U_{n,t-1}(x) - U_{\infty,t-1}(x)| \leq |U_{n,t-1}(0) - U_{\infty,t-1}(0)| + \int_0^x |U'_{n,t-1}(y) - U'_{\infty,t-1}(y)|dy.$$

As $\int_0^x |U'_{n,t-1}(y) - U'_{\infty,t-1}(y)|dy \leq NC_{t-1}(N)g(n)$, it remains to estimate the first term on the right-hand side.

$$\begin{aligned}
|U_{n,t-1}(0) - U_{\infty,t-1}(0)| &\leq |E(U_{n,t}(\tilde{\xi}_{n,t}(0)\Delta S_t)|\mathcal{F}_{t-1}) - E(U_{\infty,t}(\tilde{\xi}_{\infty,t}(0)\Delta S_t)|\mathcal{F}_{t-1})| \\
&\leq \sup_{y \in [-\hat{M}_t(0)R, \hat{M}_t(0)R]} |U_{n,t}(y) - U_{\infty,t}(y)| \\
&\quad + E(U'_{\infty,t}(-\hat{M}_t(0)R)|\tilde{\xi}_{n,t}(0) - \tilde{\xi}_{\infty,t}(0)|R|\mathcal{F}_{t-1}) \\
&\leq \tilde{C}_t(\hat{M}_t(0)R)g(n) + U'_\infty(-H_t(\hat{M}_t(0)R))K_t(0)Rg(n),
\end{aligned}$$

using (6) and (17) for $s = t-1$, the fact that $U'_{\infty,t}$ is nonincreasing, (20), (10) and (19) for $s = t$. Define

$$\tilde{C}_{t-1}(N) =: \tilde{C}_t(\hat{M}_t(0)R) + U'_\infty(-H_t(\hat{M}_t(0)R))K_t(0)R + NC_{t-1}(N),$$

this completes the induction step and hence the proof. □

6 Proof of the main results

Proof of Theorem 2.11. Suppose that the Theorem fails and we have $\psi^*_{n,t}(z) \nrightarrow \psi^*_{\infty,t}(z)$ for some t and $z \in \mathbb{R}$. We may and will suppose $\psi^*_{n,s}(z) \to \psi^*_{\infty,s}(z)$ a.s. $1 \leq s < t$. The $\psi^*_{n,t}(z)$, $n \in \mathbb{N}$ are uniformly bounded by (12), hence an argument similar to that of Lemma 2 in Kabanov and Stricker (2001) provides an $\mathcal{F}_{t-1}$-measurable random subsequence $n(k)$ such that

$$\psi^*_{n(k),t}(z) \to \hat{\psi}_t \text{ a.s., } k \to \infty,$$

and $\hat{\psi}_t$ differs from $\psi^*_{\infty,t}(z)$ on a set $A \in \mathcal{F}_{t-1}$ of positive measure. Define $\hat{\psi}_s := \psi^*_{\infty,s}(z)$ for $s < t$. Then $V^{z,\psi^*_\infty(z)}_{t-1} = V^{z,\hat{\psi}}_{t-1}$ and by (6) and (11),

$$\begin{aligned}
U_{\infty,t-1}(V^{z,\psi^*_\infty(z)}_{t-1}) &= E(U_{\infty,t}(V^{z,\psi^*_\infty(z)}_{t-1} + \tilde{\xi}_{\infty,t}(V^{z,\psi^*_\infty(z)}_{t-1})\Delta S_t)|\mathcal{F}_{t-1}) \\
&= E(U_{\infty,t}(V^{z,\psi^*_\infty(z)}_t)|\mathcal{F}_{t-1}).
\end{aligned}$$

As Assumption 2.3 holds, the maximizer is unique (see Proposition 4.7) so on A we obtain

$$E(U_{\infty,t}(V^{z,\hat{\psi}}_t)|\mathcal{F}_{t-1}) < E(U_{\infty,t}(V^{z,\psi^*_\infty(z)}_t)|\mathcal{F}_{t-1}). \tag{26}$$

Then,

$$|E(U_{n(k),t}(V_t^{z,\psi^*_{n(k)}})|\mathcal{F}_{t-1}) - E(U_{\infty,t}(V_t^{z,\hat{\psi}})|\mathcal{F}_{t-1})| \leq$$
$$E(|U_{n(k),t}(V_t^{z,\psi^*_{n(k)}}) - U_{\infty,t}(V_t^{z,\psi^*_{n(k)}})||\mathcal{F}_{t-1}) + E(|U_{\infty,t}(V_t^{z,\psi^*_{n(k)}}) - U_{\infty,t}(V_t^{z,\hat{\psi}})||\mathcal{F}_{t-1}).$$

By Corollaries 4.6, 5.1 and Lebesgue's theorem, the first term is $o(1)$. As $\psi^*_{n(k),s}(z) \to \hat{\psi}_s$, $s \leq t$; $V_t^{z,\psi^*_{n(k)}} \to V_t^{z,\hat{\psi}}$, continuity of $U_{\infty,t}$, Corollary 4.6 and Lebesgue's theorem imply that the second term is also $o(1)$.

Using Corollaries 4.6, 5.1 and continuity of $U_{\infty,t-1}$, we can also prove that

$$U_{n(k),t-1}(V_{t-1}^{z,\psi^*_{n(k)}}) = E(U_{n(k),t}(V_t^{z,\psi^*_{n(k)}})|\mathcal{F}_{t-1}) \to U_{\infty,t-1}(V_{t-1}^{z,\psi^*_\infty}) = E(U_{\infty,t}(V_t^{z,\psi^*_\infty})|\mathcal{F}_{t-1}),$$

almost surely as $k \to \infty$, so $E(U_{\infty,t}(V_t^{z,\psi^*_\infty})|\mathcal{F}_{t-1}) = E(U_{\infty,t}(V_t^{z,\hat{\psi}})|\mathcal{F}_{t-1})$, and we get a contradiction to (26).

□

Proof of Theorem 2.12. If not otherwise stated, suprema are taken on $[-N, N]$. We apply forward induction, the first step is as follows. Let $N > 0$, from (11) we have:

$$\sup_z |\psi^*_{n,1}(z) - \psi^*_{\infty,1}(z)| = \sup_z |\tilde{\xi}_{n,1}(z) - \tilde{\xi}_{\infty,1}(z)| \leq K_1(N)g(n),$$

using (19), so we can set $J_1(N) = K_1(N)$. By Theorem 4.4, Corollary 4.6, Lemma 5.2, Sublemma 5.3, Assumption 2.2 and the induction hypotheses:

$$\begin{aligned}
\sup_z |\psi^*_{n,t}(z) - \psi^*_{\infty,t}(z)| = \sup_z |\tilde{\xi}_{n,t}(V_{t-1}^{z,\psi^*_n(z)}) - \tilde{\xi}_{\infty,t}(V_{t-1}^{z,\psi^*_\infty(z)})| &\leq \\
\sup_z |\tilde{\xi}_{n,t}(V_{t-1}^{z,\psi^*_n(z)}) - \tilde{\xi}_{\infty,t}(V_{t-1}^{z,\psi^*_n(z)})| + \sup_z |\tilde{\xi}_{\infty,t}(V_{t-1}^{z,\psi^*_n(z)}) - \tilde{\xi}_{\infty,t}(V_{t-1}^{z,\psi^*_\infty(z)})| &\leq \\
K_t(F_{t-1}(N))g(n) + |V_{t-1}^{z,\psi^*_n(z)} - V_{t-1}^{z,\psi^*_\infty(z)}| \sup_{y\in[-F_{t-1}(N),F_{t-1}(N)]} |\tilde{\xi}'_{\infty,t}(y)| &\leq \\
K_t(F_{t-1}(N))g(n) + \frac{L_t(F_{t-1}(N) + \hat{M}_t(N)R)R}{\ell_t(F_{t-1}(N) + \hat{M}_t(N)R)\beta^2\kappa} g(n)R \sum_{j=1}^{t-1} J_j(N) =: J_t(N)g(n).&
\end{aligned}$$

The convergence rate of $u_n(G, x) = U_{n,0}(x)$ follows from (20). □

Proof of Theorem 2.15. Theorem 6.2 of Rásonyi and Stettner (2005) shows that $Q_n(z)$ is indeed an equivalent martingale measure. By Scheffé's theorem it suffices to establish almost sure convergence of $dQ_n(z)/dP$ and this will imply convergence in the total variation norm as well as (4). To see almost sure convergence we proceed as follows:

$$\begin{aligned}
|U'_n(V_T^{z,\psi^*_n(z)}) - U'_\infty(V_T^{z,\psi^*_\infty(z)})| \leq\ & |U'_n(V_T^{z,\psi^*_n(z)}) - U'_\infty(V_T^{z,\psi^*_n(z)})| + \\
& + |U'_\infty(V_T^{z,\psi^*_n(z)}) - U'_\infty(V_T^{z,\psi^*_\infty(z)})|.
\end{aligned}$$

As $|V_T^{z,\psi^*_n(z)}| \leq F_T(|z|)$, Remark 2.5 implies that the first term goes to zero *a.s.* By Theorem 2.11, $V_T^{z,\psi^*_n(z)} \to V_T^{z,\psi^*_\infty(z)}$ and by continuity of U'_∞, the second term tends to 0 *a.s.* Thus

$$U'_n(V_T^{z,\psi^*_n(z)}) \to U'_\infty(V_T^{z,\psi^*_\infty(z)}), \quad n \to \infty,$$

almost surely. This sequence is bounded by $\sup_{n\in\mathbb{N}} U'_n(-F_T(|z|))$ (which is finite by Remark 2.5). Hence Lebesgue's theorem implies

$$EU'_n(V_T^{z,\psi^*_n(z)}) \to EU'_\infty(V_T^{z,\psi^*_\infty(z)}), \quad n \to \infty.$$

Now, it is easy to see that if two sequences x_n and y_n converge to x_∞ and y_∞ respectively and y_n is bounded away from 0, then x_n/y_n converges to x_∞/y_∞. This observation remains true if the convergences are at the same rate $g(n)$. We want to apply to the present case with the choice $x_n := U'_n(V_T^{z,\psi^*_n(z)})$, $y_n := Ex_n$. Indeed, $y_n \geq \inf_{n\in\mathbb{N}} U'_n(F_T(|z|)) > 0$, by convergence of the $U'_n \to U'_\infty$ and strict monotonicity of U'_∞; so we get that $dQ_n(z)/dQ \to dQ_\infty(z)/dQ$ $a.s.$

Under the additional Assumption 2.8 of Theorem 2.12, the previous estimations get more precise, indeed, for all $z \in [-N, N]$,

$$\begin{aligned}
|U'_n(V_T^{z,\psi^*_n(z)}) - U'_\infty(V_T^{z,\psi^*_\infty(z)})| &\leq \sup_{y\in[-F_T(|z|),F_T(|z|)]} |U'_n(y) - U'_\infty(y)| + \\
&\quad |V_T^{z,\psi^*_n(z)} - V_T^{z,\psi^*_\infty(z)}| \sup_{y\in[-F_T(|z|),F_T(|z|)]} |U''_\infty(y)| \\
&\leq C(F_T(N))g(n) + L(F_T(N))Rg(n)\sum_{j=1}^{T} J_j(N).
\end{aligned}$$

This proves that $x_n \to x_\infty$ at the given rate $g(n)$ and by Lebesgue's theorem the same holds for y_n. □

Proof of Theorem 2.17. Let p be any accumulation point of the sequence $p_n(G, x)$ (which is included in $[0, \|G\|_\infty]$), and let n_k be a subsequence along which

$$\lim_{k\to\infty} p_{n_k}(G, x) = p.$$

Note that

$$\begin{aligned}
|u_{n_k}(G, x + p_{n_k}(G, x)) - u_\infty(G, x + p)| &\leq |u_{n_k}(G, x + p_{n_k}(G, x)) - u_\infty(G, x + p_{n_k}(G, x))| \\
&\quad + |u_\infty(G, x + p_{n_k}(G, x)) - u_\infty(G, x + p)|.
\end{aligned}$$

The first term tends to 0 by Corollary 5.1 and the fact that $x+p_{n_k}(G, x) \in [-|x|, |x|+\|G\|_\infty]$. The second one is $o(1)$ by the continuity of $u_\infty(G, .)$ and $p_{n_k}(G, x) \to p$. Since by definition of $p_{n_k}(G, x)$,

$$u_{n_k}(G, x + p_{n_k}(G, x)) = u_{n_k}(0, x),$$

and we know from Corollary 5.1 that $u_{n_k}(0, x) \to u_\infty(0, x)$, we get that

$$u_\infty(G, x + p) = u_\infty(0, x),$$

and then necessarily $p = p_\infty(G, x)$, by definition. □

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