Convergence of utility indifference prices to the superreplication price : the whole real line case*

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Abstract

A discrete-time financial market model is considered with a sequence of investors whose preferences are described by concave strictly increasing functions defined on the whole real line. Under suitable conditions we prove that, whenever their absolute risk-aversion tends to infinity, the respective utility indifference prices of a given bounded contingent claim converge to the superreplication price. We also prove that the optimal strategies asymptotically superreplicate the claim.

Keywords: derivative pricing, utility indifference price, superreplication, utility maximization.

1 Introduction

We consider a sequence of investors indexed by n; preferences of investor n are expressed via the choice of his or her concave strictly increasing utility function U_n with $\mathrm{dom}(U_n) = \mathbb{R}$. The utility indifference price (also called Hodges-Neuberger price or reservation price) for the seller of a contingent claim has been introduced by Hodges and Neuberger (1989). It is the minimal amount a seller should add to his or her initial wealth so as to reach an optimal expected utility when delivering the claim which is greater than or equal to the one he or she would have obtained trading in the basic assets only. The superreplication price is the minimal initial wealth needed for hedging the claim without risk; this is thus a utility-free pricing concept.

We will prove that (under appropriate conditions) the convergence of utility indifference prices to the superreplication price takes place for bounded contingent claims when the absolute risk-aversion $r_n = -U_n''/U_n'$ of the respective agents tends to infinity.

Up to now, this result was known for exponential utility function or for utility functions with domain $\mathrm{dom}(U_n) = (0, \infty)$ (see Carassus and Rásonyi (2005) for this latter result and also for further references).

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To study the Hodges-Neuberger price, it is necessary to solve the corresponding utility maximization problem. As usual, the whole real line case is more difficult to treat. Surprisingly, when r_n goes to infinity, we don't need to impose the "asymptotic elasticity" condition on U_n (see Kramkov and Schachermayer (1999)) to prove the existence of optimal strategies, for n big enough.

We also prove that the optimal strategies asymptotically superreplicate the claim.

In section 2 we present the model and main results. Section 3 sums up a few facts about utility maximization. Section 4 proves the main results. The appendix contains some technical results used in the proofs.

2 Definitions, assumptions and result

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ be a discrete-time filtered probability space with time horizon $T \in \mathbb{N}$. We assume that $\mathcal{F}_0$ coincides with the family of P-zero sets. If Y is a random variable, we denote by $\sup_{\omega\in\Omega} Y$ its essential supremum (in $\mathbb{R} \cup \{\infty\}$).

Let $\{S_t,\ 0 \le t \le T\}$ be a d-dimensional adapted process representing the discounted (by some numéraire) price of d securities in a given economy. The notation $\Delta S_t := S_t - S_{t-1}$ will often be used.

Trading strategies are given by d-dimensional processes $\{\phi_t,\ 1 \le t \le T\}$ which are supposed to be predictable (*i.e.* ϕ_t is $\mathcal{F}_{t-1}$-measurable). The class of all such strategies is denoted by Φ. Denote by L^∞, L^∞_+ the sets of bounded, nonnegative bounded random variables, respectively, equipped with the supremum norm $\|\cdot\|_\infty$. Trading is assumed to be self-financing, so the value process of a portfolio $\phi \in \Phi$ is

$$V_t^{z,\phi} := z + \sum_{j=1}^{t} \langle \phi_j, \Delta S_j \rangle,$$

where z is the initial capital of the agent in consideration and $\langle\cdot,\cdot\rangle$ denotes scalar product in $\mathbb{R}^d$.

The following absence of arbitrage condition is standard:

$$(NA):\ \forall \phi \in \Phi\ (V_T^{0,\phi} \ge 0 \text{ a.s.} \Rightarrow V_T^{0,\phi} = 0 \text{ a.s.}).$$

However, we need to assume a certain strengthening of the above concept hence an alternative characterization is provided in the Proposition below. Denote by $D_t(\omega)$ the smallest affine hyperplane containing the support of the (regular) conditional distribution of ΔS_t with respect to $\mathcal{F}_{t-1}$, see Proposition 8.1 of Rásonyi and Stettner (2005) for more information about the random set D_t. Let Ξ_t denote the set of $\mathcal{F}_t$-measurable d-dimensional random variables,

$$\tilde{\Xi}_t := \{\xi \in \Xi_t :\ \xi \in D_{t+1} \text{ a.s.},\ |\xi| = 1 \text{ on } \{D_{t+1} \neq \{0\}\}\}.$$

Proposition 2.1 *(NA) holds iff there exist $\mathcal{F}_t$-measurable, strictly positive, random variables κ_t, β_t, $0 \le t \le T-1$ such that*

$$\text{ess.}\inf_{\xi\in\tilde{\Xi}_t} P(\langle \xi, \Delta S_{t+1}\rangle < -\beta_t | \mathcal{F}_t) > \kappa_t \text{ a.s. on } \{D_{t+1} \neq \{0\}\}. \tag{1}$$

Proof. The direction $(NA) \Rightarrow (1)$ is Proposition 3.3 of Rásonyi and Stettner (2005). The other direction is clear from the implication $(g) \Rightarrow (a)$ in Theorem 3 of Jacod and Shiryaev (1998). □

In Theorems 2.5 and 2.7 a strengthening of this condition will be required. Similar "uniform no arbitrage" assumptions appear in Schäl (1999,2000).

Assumption 2.2 There exist constants $\beta, \kappa > 0$ such that

$$\text{ess.} \inf_{\xi \in \tilde{\Xi}_t} P(\langle \xi, \Delta S_{t+1} \rangle < -\beta | \mathcal{F}_t) > \kappa \text{ a.s. on } \{D_{t+1} \neq \{0\}\}.$$

Fix $G \in L_+^\infty$, a random variable which will be interpreted as the payoff of some derivative security at time T.

The concept of superreplication price is formally introduced as the minimal initial wealth needed for hedging without risk the given contingent claim:

$$\pi(G) := \inf\{z \in \mathbb{R} : \ V_T^{z,\phi} \geq G \, a.s. \text{ for some } \phi \in \Phi\}.$$

We go on incorporating a sequence of agents in our model with concave utility functions U_n. The functions r_n below express the absolute risk-aversion of the respective agents.

Assumption 2.3 Suppose that $U_n : \mathbb{R} \to \mathbb{R}, \ n \in \mathbb{N}$ is a sequence of concave strictly increasing twice continuously differentiable functions such that

$$\forall x \in \mathbb{R} \quad r_n(x) := -\frac{U_n''(x)}{U_n'(x)} \to \infty, \ n \to \infty.$$

Now define

$$u_n(G, z) := \sup_{\phi \in \Phi(U_n, G, z)} EU_n(V_T^{z,\phi} - G), \tag{2}$$

where $\Phi(U_n, G, z)$ denotes the family of strategies $\phi \in \Phi$ such that $EU_n(V_T^{z,\phi} - G)$ exists. The quantity $u_n(G, z)$ represents the supremum of expected utility from initial capital z delivering a contingent claim with payoff G at the terminal date.

Definition 2.4 The utility indifference price $p_n(G, x)$ is defined as

$$p_n(G, x) = \inf\{z \in \mathbb{R} : u_n(G, x + z) \geq u_n(0, x)\}.$$

In the following Theorem, we show that due to the specific convergence of U_n (see Lemma 5.4) when n goes to infinity, we can construct the optimal strategies $\psi_n^*(z)$, for n big enough, without the usual "asymptotic elasticity" conditions. Moreover, we prove that pursuing the optimal strategies one can asymptotically super-replicate the given claim.

Theorem 2.5 *Suppose that S is bounded, Assumptions 2.2 and 2.3 hold.*
Then, there exists $N_0 \in \mathbb{N}$, such that for all $n \geq N_0$ and $z \in \mathbb{R}$, the utility maximization problem (2) *admits optimal solutions $\psi_n^*(z)$.*
Furthermore, for any $z \geq \pi(G)$,

$$\lim_{n \to \infty} P(V_T^{z,\psi_n^*(z)} \geq G) = 1.$$

Remark 2.6 For $T = 1$, we will show in Theorem 3.1 that there exists some subsequence $(n_k)_{k\geq 0}$ converging to the constant $\tilde{\psi}$ and such that $\psi^*_{n_k}(\pi(G)) \to \tilde{\psi}$ and $\tilde{\psi}$ is a superhedging strategy for G.

This result have been obtained under different conditions by Summer (2002) and Cheridito and Summer (2003). More precisely, they introduced special super replication strategies, called balanced strategies, and they showed that the distance between the set of such strategies and the optimal ones goes to zero if $\lim_n \inf_{x\in\mathbb{R}} r_n(x) = +\infty$.

One would also conjecture that the optimal strategies converge to some superreplication strategy. This is, however, false, see the example of Cheridito and Summer (2003).

We wish to find conditions on S and U_n which guarantee that $p_n(G, x)$ tends to $\pi(G)$ whenever Assumption 2.3 holds.

Theorem 2.7 *Suppose that S is bounded, Assumptions 2.2 and 2.3 hold. Then for each $x_0 \in \mathbb{R}$ the utility prices $p_n(G, x_0)$ are well-defined (for n sufficiently large) and converge to $\pi(G)$ as $n \to \infty$.*

Remark 2.8 We present a proof of these results working on the primal problem only. It is possible to prove them under very similar assumptions relaying on duality techniques. However, the latter method would necessitate that we use a class of strategies which is unnatural in the discrete-time context and would also require an analysis of the structure of equivalent martingale measures, hence we opted for the present approach.

3 Utility maximization

In this section we use a dynamic programming procedure to prove the existence of optimal strategies and to derive bounds on them. We first introduce the recursive superreplication price of any $G \in L^\infty_+$:

$$\begin{aligned} \pi_T(G) &:= G, \\ \pi_t(G) &= \operatorname{ess.inf}\{X : \sigma(X) \subset \mathcal{F}_t, \exists \phi \in \Xi_t \text{ such that} \\ &\qquad X + \langle \phi, \Delta S_{t+1}\rangle \geq \pi_{t+1}(G) \text{ a.s.}\}, \end{aligned}$$

for $0 \leq t \leq T-1$. Note that $\pi_0(G)$ can be chosen constant, by the triviality of $\mathcal{F}_0$. It is easy to see that $\pi_0(G) = \pi(G)$ (see Proposition 5.1) and that $0 \leq \pi_t(G) \leq \|G\|_\infty$.

We will also use the following sets : $\mathcal{V}_T = \{0\}$ and for $0 \leq t \leq T-1$

$$\mathcal{V}_t := \{\sum_{j=t+1}^{T} \langle \zeta_j, \Delta S_j\rangle, \quad \zeta_j \in \Xi_{j-1},\ j = t+1, \ldots, T\}. \tag{3}$$

Interestingly enough, the following Theorem provides optimal strategies (for n large enough) without the usual "asymptotic elasticity" conditions and derives bounds on them, which are crucial in the sequel. Convergence properties of the value function are also established here.

Theorem 3.1 *Suppose that S is bounded and Assumptions 2.2, 2.3 and $U_n(0) = 0$, $U_n'(0) = 1$ hold for all $n \in \mathbb{N}$. Then there exist constants N_s, $0 \leq s \leq T$ such that for $n \geq N_s$ the random functions $U_{n,s}$ below are well defined and finite:*

$$\begin{aligned} U_{n,T}(x) &:= U_n(x - G), \\ U_{n,s}(x) &:= \operatorname{ess.}\sup_{\xi\in\Xi_s} E(U_{n,s+1}(x + \langle \xi, \Delta S_{s+1}\rangle)|\mathcal{F}_s), \quad 0 \leq s \leq T-1, \end{aligned}$$

we also have for all $x, y \in \mathbb{R}$

$$-\infty < EU_{n,s}(x + y\Delta S_t) < \infty. \tag{4}$$

The functions $U_{n,s}$ have almost surely concave and increasing continuously differentiable versions. For all $1 \leq s \leq T$, $n \geq N_s$ and $x \in \mathbb{R}$, there exists $\tilde{\xi}_{n,s}(x) \in \Xi_{s-1}$ such that $\tilde{\xi}_{n,s} \in D_s$ a.s. and

$$\begin{aligned} U_{n,s-1}(x) &= E(U_{n,s}(x + \langle \tilde{\xi}_{n,s}(x), \Delta S_s \rangle)|\mathcal{F}_{s-1}), & (5)\\ U'_{n,s-1}(x) &= E(U'_{n,s}(x + \langle \tilde{\xi}_{n,s}(x), \Delta S_s \rangle)|\mathcal{F}_{s-1}). & (6) \end{aligned}$$

For all $1 \leq s \leq T$, there exist nondecreasing functions M_s and $\hat{M}_s : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $n \geq N_s$ and $x \in \mathbb{R}$:

$$\begin{aligned} |\tilde{\xi}_{n,s}(x)| &\leq \hat{M}_s(x), & (7)\\ U_n(x - M_s(|x|)) &\leq U_{n,s}(x) \leq U_n(x + M_s(|x|)). & (8) \end{aligned}$$

Furthermore, for all $0 \leq s \leq T$ and $x \in \mathbb{R}$

$$\sup_{\omega \in \Omega} U_{n,s}(x) \to 0 \text{ on } \{x \geq \pi_s(G)\}. \tag{9}$$

For all $0 \leq s \leq T$ and $\varepsilon > 0$,

$$U_{n,s}(\pi_s(G) - \varepsilon) \to -\infty \tag{10}$$

almost surely.
For all $1 \leq s \leq T$, there exist some random subsequence $(\sigma_{k,s})_{k \geq 0}$ and an $\mathcal{F}_{s-1}$-measurable random variable $\tilde{\xi}_s$ such that $\tilde{\xi}_{\sigma_{k,s},s}(\pi_{s-1}(G)) \to \tilde{\xi}_s$ a.s and $(\tilde{\xi}_s)_{1 \leq s \leq T}$ is a superhedging strategy for G.

For all $n \geq N_0$, $z \in \mathbb{R}$ the utility maximization problems

$$EU_n(V_T^{z,\psi} - G) \to \max., \ \psi \in \Phi(U_n, G, z),$$

admit optimal solutions $\psi_n^*(z)$ given by

$$\psi_{n,1}^*(z) := \tilde{\xi}_{n,1}(z), \quad \psi_{n,t+1}^*(z) := \tilde{\xi}_{n,t+1}(z + \sum_{k=1}^{t} \langle \psi_{n,k}^*(z), \Delta S_k \rangle) \tag{11}$$

There exists nondecreasing functions $\Upsilon_t : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $n \in \mathbb{N}$, $z \in \mathbb{R}$

$$|\psi_{n,t}^*(z)| \leq \Upsilon_t(|z|). \tag{12}$$

and the value functions of the optimization problems are finite, i.e.

$$u_n(G, z) = U_{n,0}(z) < \infty.$$

Corollary 3.2 *Under the conditions of Theorem 3.1, there exist nondecreasing functions $F_t : \mathbb{R}_+ \to \mathbb{R}_+,\ 0 \leq t \leq T$ such that for all $n \in \mathbb{N}$*

$$|V_t^{z,\psi_n^*(z)}| \leq F_t(|z|) \text{ a.s.}$$

for the optimal strategies $\psi_n^*(z)$ constructed in the previous Theorem.

Proof of Corollary 3.2. Indeed, define $F_t(u) := u + R\left[\sum_{j=1}^{t} \Upsilon_j(u)\right]$. $\square$

Proof of Theorem 3.1. Suppose $d = 1$ for notational simplicity and let R denote a constant bound for the process $|\Delta S|$.

Results of Rásonyi and Stettner (2005) will be used, namely Propositions 4.4 to 4.6, Propositions 4.9, 4.10 and 6.5. Note that those propositions do not rely on the "asymptotic elasticity" property which is crucial to prove the existence of optimal strategy in the cited paper. To achieve the same goal without this hypothesis, we will carry out the estimations in a different way.

We shall apply backward induction to prove (4) to (10). First for $s = T$, set $N_T = 0$, (4) and (8) are trivial ; (9) and (10) hold by Lemma 5.4. From Proposition 4.4 and Proposition 6.5 of Rásonyi and Stettner (2005), $U_{n,T-1}$ have almost surely concave and increasing continuously differentiable versions. Finally, (5), (6) and (7) will follow just like in the induction step below.

Let us proceed supposing that the induction hypotheses hold for $s \geq t+1$. We get from (7) for $s = t+1$ that

$$x + \tilde{\xi}_{n,t+1}(x)\Delta S_{t+1} \in [x - \hat{M}_{t+1}(|x|)R, x + \hat{M}_{t+1}(|x|)R],$$

and from (8) for $s = t+1$

$$U_{n,t+1}(x + \hat{M}_{t+1}(|x|)R) \leq U_n\left(x + \hat{M}_{t+1}(|x|)R + M_{t+1}(|x| + \hat{M}_{t+1}(|x|)R)\right)$$

because M_{t+1} and U_n are nondecreasing. Also

$$U_{n,t+1}(x - \hat{M}_{t+1}(|x|)R) \geq U_n\left(x - \hat{M}_{t+1}(|x|)R - M_{t+1}(|x| + \hat{M}_{t+1}(|x|)R)\right).$$

Defining

$$M_t(u) := \hat{M}_{t+1}(u)R + M_{t+1}(u + \hat{M}_{t+1}(u)R),\ u \in \mathbb{R}_+,$$

M_t is nondecreasing as $\hat{M}_{t+1}$ and M_{t+1} are. Using (5) for $s = t+1$ and the fact that $U_{n,t+1}$ is nondecreasing, we get that almost surely

$$U_n(x - M_t(|x|)) \leq U_{n,t}(x) \leq U_n(x + M_t(|x|)),$$

showing (8) for $s = t$. Moreover, as S is bounded, it is easy to see that (4) holds true for $s = t-1$. So we can again apply Proposition 4.4 of Rásonyi and Stettner (2005) and $U_{n,t-1}$ have almost surely concave and increasing versions.

It turns out that $U_{n,t}(\pi_t(G))$ is nonnegative. Indeed, take a strategy $\hat{\phi}$ such that $\pi_t(G) + \hat{\phi}\Delta S_{t+1} \geq \pi_{t+1}(G)$ almost surely. Then we have

$$\begin{aligned}
U_{n,t}(\pi_t(G)) &= \operatorname{ess.sup}_{\phi} E(U_{n,t+1}(\pi_t(G) + \phi\Delta S_{t+1})|\mathcal{F}_t) \\
&\geq E(U_{n,t+1}(\pi_t(G) + \hat{\phi}\Delta S_{t+1})|\mathcal{F}_t) \geq E(U_{n,t+1}(\pi_{t+1}(G))|\mathcal{F}_t) \\
&\geq E(E(U_{n,t+2}(\pi_{t+2}(G))|\mathcal{F}_{t+1})|\mathcal{F}_t) \geq \ldots \geq E(U_{n,T}(G)|\mathcal{F}_t) \\
&\geq U_n(0) = 0.
\end{aligned}$$

Then for $\{x \geq \pi_t(G)\}$, we get from (8) for $s = t$ that

$$\begin{aligned}
0 \leq U_{n,t}(x) &\leq U_{n,t}(x + \|G\|_\infty) \\
&\leq U_n(x + \|G\|_\infty + M_t(x + \|G\|_\infty)).
\end{aligned}$$

Thus $\sup_{\omega\in\Omega} U_{n,t}(\pi_t(G)) \to 0$ by Lemma 5.4 and we get that (9) holds for $s = t$.

Now we establish the existence of optimal strategies. To this end, we need two auxiliary results.

Lemma 3.3 *Under the induction hypothesis, for each* $0 \le t \le T$ *and* $y > \|G\|_\infty$,

$$\sup_{\omega\in\Omega} U'_{n,t}(y) \to 0.$$

Proof. By the induction hypotheses $U_{n,t}$ is continuously differentiable and we can write that *a.s.*

$$U_{n,t}(\|G\|_\infty) + \int_{\|G\|_\infty}^{y} U'_{n,t}(w)dw = U_{n,t}(y).$$

We have, by monotonicity of the derivative,

$$U'_{n,t}(y)[y - \|G\|_\infty] \le \int_{\|G\|_\infty}^{y} U'_{n,t}(w)dw \le |U_{n,t}(y)| + |U_{n,t}(\|G\|_\infty)|,$$

so the statement follows by (9) for $s = t$ and $\pi_t(G) \le \|G\|_\infty$. □

Proposition 3.4 *Assume the induction hypothesis and let* $x \in \mathbb{R}$. *There exist an increasing function* $\hat{M}_t(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ *such that for any* $\xi \in \Xi_t$, $\xi \in D_{t+1}$ *a.s. satisfying* $|\xi| > \hat{M}_t(|x|)$,

$$E(U_{n,t}(x + \xi\Delta S_t)|\mathcal{F}_{t-1}) < E(U_{n,t}(x)|\mathcal{F}_{t-1}), \tag{13}$$

for $n > N_t$, *where* N_t *is a suitable constant.*

Proof. Fix $x \in \mathbb{R}$ and suppose that $n \ge N_{t+1}$. Let $\xi \in \Xi_t$, $\xi \in D_{t+1}$ *a.s* satisfying

$$|\xi| > \frac{x}{\beta}. \tag{14}$$

Let V_n be the "optimal continuation" from $x+\xi\Delta S_t$ for the utility $U_{n,t}$. To write it formally, we introduce the following

$$\begin{aligned}
\phi_{n,t+1} &:= \tilde{\xi}_{n,t+1}(x + \xi\Delta S_t),\\
\phi_{n,j+1} &:= \tilde{\xi}_{n,j+1}(x + \xi\Delta S_t + \sum_{k=t+1}^{j} \phi_{n,k}\Delta S_k) \quad \text{for } j = t+1, \dots, T-1,\\
V_n &:= \sum_{k=t+1}^{T} \phi_{n,k}\Delta S_k.
\end{aligned}$$

It is clear that $V_n \in \mathcal{V}_t$. By (5) for $s = t+1, \dots, T$ and Proposition 4.10 of Rásonyi and Stettner (2005), we obtain that,

$$\begin{aligned}
E(U_{n,t}(x + \xi\Delta S_t)|\mathcal{F}_{t-1}) &= E(U_n(x + \xi\Delta S_t + V_n - G)|\mathcal{F}_{t-1})\\
&\le E(U_n(x - \beta|\xi|)I_{\{\xi\Delta S_t + V_n < -\beta|\xi|\}}|\mathcal{F}_{t-1}) + E(U_{n,t}(|x| + 2\|G\|_\infty + |\xi|R)|\mathcal{F}_{t-1}).
\end{aligned}$$

Note first that $U_n(x - \beta|\xi|) \le U_n(0) = 0$ from (14) and also that

$$\begin{aligned}
U_{n,t}(|x| + 2\|G\|_\infty + |\xi|R) &\ge E(U_{n,t+1}(|x| + 2\|G\|_\infty + |\xi|R)|\mathcal{F}_t)\\
&\ge E(U_{n,T}(|x| + 2\|G\|_\infty + |\xi|R)|\mathcal{F}_t) \ge U_n(0) = 0.
\end{aligned}$$

Thus by Proposition 5.2, we obtain

$$E(U_{n,t}(x+\xi\Delta S_t)|\mathcal{F}_{t-1}) \leq \kappa^{T-t+1}U_n(x-\beta|\xi|) + E(U_{n,t}(|x|+2\|G\|_\infty+|\xi|R)|\mathcal{F}_{t-1}). \quad (15)$$

Continue the estimation of the first term of the right-hand side of inequality (15):

$$\begin{aligned}
\kappa^{T-t+1}U_n(x-\beta|\xi|) &= \frac{\kappa^{T-t+1}}{2}U_n(x-\beta|\xi|) + \frac{\kappa^{T-t+1}}{2}U_n(x-\beta|\xi|) \\
&\leq U_n(\frac{\kappa^{T-t+1}}{2}(x-\beta|\xi|)) + \frac{\kappa^{T-t+1}}{2}\left[U_n(x-\frac{\beta|\xi|}{2}) - \frac{\beta|\xi|}{2}U_n'(x-\frac{\beta|\xi|}{2})\right],
\end{aligned}$$

where we used concavity of U_n. Choose ξ such that, in addition to (14), both

$$\begin{aligned}
x-\frac{\beta|\xi|}{2} &< -1, & (16)\\
\frac{\kappa^{T-t+1}}{2}(x-\beta|\xi|) &< x-M_t(|x|), & (17)
\end{aligned}$$

hold. For the estimation of the second term of the right-hand side of (15), we use concavity of $U_{n,t}$ and (8) for $s=t$ to see that

$$\begin{aligned}
U_{n,t}(|x|+2\|G\|_\infty+|\xi|R) &\leq U_{n,t}(|x|+2\|G\|_\infty) + U_{n,t}'(|x|+2\|G\|_\infty)|\xi|R \\
&\leq U_n(|x|+2\|G\|_\infty+M_t(|x|+2\|G\|_\infty)) + U_{n,t}'(2\|G\|_\infty)|\xi|R \\
&\leq |x|+2\|G\|_\infty+M_t(|x|+2\|G\|_\infty) + U_{n,t}'(2\|G\|_\infty)|\xi|R,
\end{aligned}$$

as $U_n(0)=0$ and $U_n'(0)=1$. So we get that by (16) and (17) that

$$\begin{aligned}
E(U_{n,t}(x+\xi\Delta S_t)|\mathcal{F}_{t-1}) &< U_n(x-M_t(|x|)) \\
&\quad +|\xi|\left[U_{n,t}'(2\|G\|_\infty)R - \frac{\beta\kappa^{T-t+1}}{4}U_n'(-1)\right] \\
&\quad +|x|+2\|G\|_\infty+M_t(|x|+2\|G\|_\infty) + (\kappa^{T-t+1}/2)U_n(-1)].
\end{aligned}$$

Here the first term is $\leq U_{n,t}(x)$ by (8). Note also that $U_n(-1)\leq 0$. Now by Lemma 3.3 and Lemma 5.4, there exists some $N_t \geq N_{t+1}$ such that for $n \geq N_t$,

$$U_{n,t}'(2\|G\|_\infty)R - \frac{\beta\kappa^{T-t+1}}{4}U_n'(-1) \leq -1.$$

Thus

$$E(U_{n,t}(x+\xi\Delta S_t)|\mathcal{F}_{t-1}) < E(U_{n,t}(x)|\mathcal{F}_{t-1}) - |\xi| + |x| + 2\|G\|_\infty + M_t(|x|+2\|G\|_\infty).$$

Choosing $\hat{M}_t$ so large that if $|\xi| > \hat{M}_t(|x|)$, then $|\xi| > |x|+2\|G\|_\infty+M_t(|x|+2\|G\|_\infty)$, (14), (16) and (17) are satisfied. □

Using Proposition 3.4 and a compactness argument, we are now able to prove that a bounded optimal strategy $\tilde{\xi}_{n,t}(x)$ exists for $x \in [0,1]$ (which implies the existence on the whole real line).

Take a jointly measurable sequence $\{\xi_{n,t}^k(x,\omega), k \in \mathbb{N}\}$ attaining the essential supremum in the definition of $U_{n,t}$ (constructed in Lemma 4.5 of Rásonyi and Stettner (2005))

By Proposition 4.6 of the same paper we may suppose that $\xi^k_{n,t}(x) \in D_t$ *a.s.* Let $A = \{|\xi^k_{n,t}(x)| > \hat{M}_t(x)\} \in \mathcal{F}_{t-1}$. From Proposition 3.4,

$$\begin{aligned} E(U_{n,t}(x+\xi^k_{n,t}(x)\Delta S_t)|\mathcal{F}_{t-1}) &\leq I_A E(U_{n,t}(x)|\mathcal{F}_{t-1}) + I_{A^c} E(U_{n,t}(x+\xi^k_{n,t}(x)\Delta S_t)|\mathcal{F}_{t-1}) \\ &\leq E(U_{n,t}(x+\xi^k_{n,t}(x)I_{A^c})\Delta S_t)|\mathcal{F}_{t-1}), \end{aligned}$$

with strict inequality on A. Call

$$\zeta^k_{n,t}(x) := \xi^k_{n,t}(x) I_{A^c},$$

by optimality we can replace the sequence $\{\xi^k_{n,t}(x), k \in \mathbb{N}\}$ by $\{\zeta^k_{n,t}(x), k \in \mathbb{N}\}$.

As $\hat{M}_t(.)$ is increasing,

$$|\zeta^k_{n,t}(x)| \leq \hat{M}_t(1),$$

and Lemma 2 of Kabanov-Stricker (2001) proves the existence of a random subsequence σ_k such that of $\zeta^{\sigma_k}_{n,t}(x)$ converges to some limit called $\tilde{\xi}_{n,t}(x)$. We have $\tilde{\xi}_{n,t}(x) \in \Xi_{t-1}$ and $\in D_t$ *a.s.*

Now by Fatou Lemma (which applied because of (8) for $s = t$),

$$\begin{aligned} U_{n,t-1}(x) &= \limsup_{k\to\infty} E(U_{n,t}(x+\zeta^{\sigma_k}_{n,t}(x)\Delta S_t)|\mathcal{F}_{t-1}) \\ &\leq E(U_{n,t}(x+\tilde{\xi}_{n,t}(x)\Delta S_t)|\mathcal{F}_{t-1}), \end{aligned}$$

which proves (5) and (7) for $s = t$.
Moreover, we get from Proposition 6.5 of Rásonyi and Stettner (2005) that $U_{n,t-1}$ has almost surely continuously differentiable versions and (6) is satisfied for $s = t$.

To prove (10) for $s = t$, we first recall that by (5) for $s = t, \ldots, T$ and Proposition 4.10 of Rásonyi and Stettner (2005)

$$U_{n,t}(\pi_t(G) - \varepsilon) = E(U_n(\pi_t(G) - \varepsilon + V_n - G)|\mathcal{F}_t)$$

for some $V_n \in \mathcal{V}_t$ (the "optimal continuation" from $\pi_t(G) - \varepsilon$). Now apply Lemma 5.3 and (8) to obtain the estimation

$$\begin{aligned} U_{n,t}(\pi_t(G) - \varepsilon) &\leq E(U_n(-\varepsilon/2) I_{\pi_t(G)-\varepsilon/2+V_n<G}|\mathcal{F}_t) + U_n(\|G\|_\infty + M_t(\|G\|_\infty)) \\ &\leq Z U_n(-\varepsilon/2) + U_n(\|G\|_\infty + M_t(\|G\|_\infty)). \end{aligned}$$

Here the second term tends to 0 and the first to $-\infty$, by Lemma 5.4, proving (10).

Now we turn to the proof of the convergence of $\tilde{\xi}_{n,t}(\pi_t(G))$. Suppose that there exists some K such that for $k \geq K$,

$$P(\pi_t(G) + \tilde{\xi}_{n,t+1}(\pi_t(G))\Delta S_{t+1} \leq \pi_{t+1}(G) - 1/k|\mathcal{F}_t) > 1/k.$$

Then as

$$U_{n,t}(\pi_t(G)) = E(U_{n,t+1}(\pi_t(G) + \tilde{\xi}_{n,t+1}(\pi_t(G))\Delta S_{t+1})|\mathcal{F}_t).$$

We get that

$$U_{n,t}(\pi_t(G)) \leq 1/k U_{n,t+1}(\pi_{t+1}(G) - 1/k) + U_n(\|G\|_\infty + M_t(\|G\|_\infty)).$$

Here the second term of the right-hand side tends to 0 by Lemma 5.4 and the first to $-\infty$ by (10) for $s = t + 1$, contradicting (9) for $s = t$. Thus there exists some random subsequence $(\sigma_k)_{k\geq 0}$ such that

$$P(\pi_t(G) + \tilde{\xi}_{\sigma_k,t+1}(\pi_t(G))\Delta S_{t+1} \leq \pi_{t+1}(G) - 1/k|\mathcal{F}_t) \leq 1/k.$$

From (7) for $s = t + 1$ and Lemma 2 of Kabanov and Stricker (2001), there exists a random, $\mathcal{F}_t$-measurable, subsequence $(\sigma_k)_{k\geq 0}$ (for which we keep the same notation) and an $\mathcal{F}_t$-measurable random variable $\tilde{\xi}_{t+1}$ such that $\tilde{\xi}_{\sigma_k,t+1}(\pi_t(G)) \to \tilde{\xi}_{t+1}$ $a.s.$

Let

$$\begin{aligned} A &= \{\pi_t(G) + \tilde{\xi}_{t+1}\Delta S_{t+1} \leq \pi_{t+1}(G)\} \\ A_k &= \{\pi_t(G) + \tilde{\xi}_{\sigma_k,t+1}(\pi_t(G))\Delta S_{t+1} \leq \pi_{t+1}(G) - 1/k\}. \end{aligned}$$

Then it is easy see that $A \subset \liminf_k A_k$ and by Fatou lemma,

$$P(A|\mathcal{F}_t) \leq P(\liminf_k A_k|\mathcal{F}_t) \leq \liminf_k P(A_k|\mathcal{F}_t).$$

So we conclude that

$$P(\pi_t(G) + \tilde{\xi}_{t+1}\Delta S_{t+1} \leq \pi_{t+1}(G)|\mathcal{F}_t) = 0.$$

Summing up all this inequalities, $a.s$

$$\pi(G) + \sum_{t=1}^{T} \tilde{\xi}_t \Delta S_t \geq G,$$

and $(\tilde{\xi}_s)_{1\leq s\leq T}$ is a superhedging strategy for G

By induction, it is easy to see from (7) that (12) holds with

$$\Upsilon_1(u) = \hat{M}_1(u) \text{ and } \Upsilon_{t+1}(u) = \hat{M}_{t+1}\left(u + R\sum_{s=1}^{t} \Upsilon_s(u)\right).$$

It remains to prove that the strategies defined by (11) are optimal.

Just like in Proposition 5.3 of Rásonyi and Stettner (2005), we obtain that for any trading strategy $\psi \in \Phi(U_n, G, z)$:

$$E(U_n(V_T^{z,\psi})|\mathcal{F}_0) \leq U_{n,0}(z) = E(U_n(V_T^{z,\psi_n^*(z)})|\mathcal{F}_0).$$

As $U_{n,0}(z)$ is finite and $\mathcal{F}_0$ is trivial one gets that $u_n(G, z) = U_{n,0}(z) < \infty$ and for all $\psi \in \Phi(U_n, G, z)$, $E(U_n(V_T^{z,\psi})) \leq E(U_n(V_T^{z,\psi_n^*(z)})) = u_n(G, z) < \infty$. Thus $\psi_n^*(z)$ is the solution of

$$EU_n(V_T^{z,\psi}) \to \max., \ \psi \in \Phi(U_n, G, z).$$

and the value functions of the optimization problems are finite, $i.e.$ $u_n(G, z) = U_{n,0}(z) < \infty$. □

4 Proof of the main results

Proof of Theorem 2.5. By taking $\frac{U_n(x)}{U_n'(0)} - \frac{U_n(0)}{U_n'(0)}$ instead of $U_n(x)$ one may assume $U_n(0) = 0$, $U_n'(0) = 1$; this does not affect the validity of Assumption 2.3 and does not change optimal strategies either. Suppose that $n \geq N_0$, then from Theorem 3.1, item (11), we get the existence of the optimal strategies $\psi_n^*(z)$.

If the second part of Theorem 2.5 did not hold, for some $\varepsilon > 0$ one would have

$$P(V_T^{z,\psi_n^*(z)} - G \leq -\varepsilon) \geq \varepsilon$$

(along a subsequence). Then by Corollary 3.2,

$$u_n(G,z) = EU_n(V_T^{z,\psi_n^*(z)} - G) \leq U_n(-\varepsilon)\varepsilon + U_n(F_T(z)).$$

Here the second term of the right-hand side tends to 0, the first to $-\infty$, which is nonsense as $u_n(G,z) \to 0$ by Theorem 3.1. □

Proof of Theorem 2.7. Fix $x_0 \in \mathbb{R}$. Suppose that $n \geq N_0$, then $p_n(G,x_0)$ is well defined. Consider the sequence $V_n(x) := \frac{U_n(x+x_0)}{U_n'(x_0)} - \frac{U_n(x_0)}{U_n'(x_0)}$. Then $V_n(0) = 0$, $V_n'(0) = 1$ and Assumption 2.3 is still valid. We will prove that the corresponding utility prices $p_{V_n}(G,0)$ tend to $\pi(G)$. As obviously $p_{U_n}(G,x_0) = p_{V_n}(G,0)$, this will implied the Theorem.

From now we denote $p_{V_n}(G,0)$ by $p_n(G,0)$. One may easily check (see the proof of Theorem 2.6 in Carassus and Rásonyi (2005)) that

$$p_n(G,0) \leq \pi(G).$$

Now it remains to prove

$$\liminf_{n\to\infty} p_n(G,0) \geq \pi(G). \tag{18}$$

Suppose that this fails, *i.e.* for some $\eta > 0$ and some subsequence n_k

$$p_{n_k}(G,0) \leq \pi(G) - \eta$$

holds, for all $k \in \mathbb{N}$. By Definition 2.4,

$$u_{n_k}(G, \pi(G) - \eta) \geq u_{n_k}(0,0).$$

The liminf of the right-hand side is nonnegative :

$$\liminf_{n\to\infty} u_n(0,0) \geq \liminf_{n\to\infty} U_n(0) = 0.$$

The left-hand side tends to $-\infty$ by item (10) of Theorem 3.1 for $s = 0$ and this contradiction proves (18). □

5 Appendix

We start this section with an alternative characterization of $\pi_t(G)$.

Proposition 5.1

$$\pi_t(G) = \text{ess.}\inf\{Y,\ \exists V \in \mathcal{V}_t :\ Y + V \geq G \text{ a.s.}\}. \tag{19}$$

In particular,

$$\pi_0(G) = \pi(G).$$

Proof. Let prove it by induction; the case $t = T$ is trivial. Suppose that the proposition holds for $t+1$. Let Y such that there exists $V = \sum_{k=t+1}^{T} \langle \phi_k, \Delta S_k \rangle \in \mathcal{V}_t$ such that $Y + V \geq G$ a.s. As $\sum_{k=t+2}^{T} \langle \phi_k, \Delta S_k \rangle \in \mathcal{V}_{t+1}$, $Y + \langle \phi_{t+1}, \Delta S_{t+1} \rangle \geq \pi_{t+1}(G)$ $a.s$, by the induction hypothesis for $t+1$. Now by definition of $\pi_t(G)$ we get that $Y \geq \pi_t(G)$ and

$$\pi_t(G) \leq \text{ess.}\inf\{Y,\ \exists V \in \mathcal{V}_t :\ Y + V \geq G \text{ a.s.}\}.$$

Conversely, fix $\varepsilon > 0$, by definition of $\pi_k(G)$, there exists $\{\phi_k,\ t \leq k \leq T-1\}$ such that $\phi_k \in \Xi_{k-1}$ and

$$\pi_k(G) + \frac{\varepsilon}{T-t} + \langle \phi_{k+1}, \Delta S_{k+1} \rangle \geq \pi_{k+1}(G) \text{ a.s.}$$

Summing over all $k = t, \ldots, T-1$,

$$\pi_t(G) + \varepsilon + \sum_{k=t+1}^{T} \langle \phi_k, \Delta S_k \rangle \geq G \text{ a.s.}$$

follows and therefore

$$\pi_t(G) + \varepsilon \geq \text{ess.}\inf\{Y,\ \exists V \in \mathcal{V}_t :\ Y + V \geq G \text{ a.s.}\},$$

so letting $\varepsilon \to 0$ achieves the proof of the Proposition. $\square$

We now deduce two consequences of Assumption 2.2:

Proposition 5.2 *For each* $1 \leq t \leq T$ *and* $\xi \in \tilde{\Xi}_{t-1}$,

$$\begin{aligned} \text{ess.}\inf_{V \in \mathcal{V}_t} P(\langle \xi, \Delta S_t \rangle + V < -\beta | \mathcal{F}_{t-1}) &\geq \kappa^{T-t+1}, \\ \text{ess.}\inf_{V \in \mathcal{V}_t} P(V \leq 0 | \mathcal{F}_{t-1}) &\geq \kappa^{T-t}. \end{aligned}$$

Proof. By backward induction. The step $t = T$ is a direct consequence of Assumption 2.2. Now suppose that the statements are shown for $t+1$, let us prove them for t. Let $V \in \mathcal{V}_t$ and $\xi \in \Xi_{t-1}$,

$$\begin{aligned} P(\langle \xi, \Delta S_t \rangle + V < -\beta | \mathcal{F}_{t-1}) &\geq P(\langle \xi, \Delta S_t \rangle < -\beta,\ V < 0 | \mathcal{F}_{t-1}) \\ &= E[E[I_{\{V \leq 0\}} | \mathcal{F}_t] I_{\{\langle \xi, \Delta S_t \rangle < -\beta\}} | \mathcal{F}_{t-1}] \\ &\geq \kappa^{T-t} \times \kappa, \end{aligned}$$

by Assumption 2.2 and the induction hypotheses. Similarly, for any $V \in \mathcal{V}_t$,

$$\begin{aligned} P(V \leq 0 | \mathcal{F}_{t-1}) &\geq P(\langle \zeta_t, \Delta S_t \rangle \leq 0,\ V - \langle \zeta_t, \Delta S_t \rangle \leq 0 | \mathcal{F}_{t-1}) \\ &= E[E[I_{\{V - \langle \zeta_t, \Delta S_t \rangle \leq 0\}} | \mathcal{F}_t] I_{\{\langle \zeta_t, \Delta S_t \rangle \leq -\beta\}} | \mathcal{F}_{t-1}] \\ &\geq \kappa^{T-t-1} \times \kappa. \end{aligned}$$

$\square$

Lemma 5.3 *For all* $\varepsilon > 0$, *we have*

$$Z := \text{ess.}\inf_{V \in \mathcal{V}_t} P(\pi_t(G) - \varepsilon/2 + V < G | \mathcal{F}_t) > 0,$$

almost surely.

Proof of Lemma. Consider the set of all random variables L^0 equipped with the topology of convergence in probability, L^0_+ denotes the set of nonnegative random variables. Suppose that the statement fails and $A := \{Z = 0\} \in \mathcal{F}_t$ has positive probability. Define the set

$$Q := \{X \in L^0 : X = \pi_t(G) - \varepsilon/2 + V \text{ for some } V \in \mathcal{V}_t\} \subset L^0.$$

Then there are $V_n \in \mathcal{V}_t$ such that for

$$B_n := \{\pi_t(G) - \varepsilon/2 + V_n \geq G\}$$

one has $P(B_n|\mathcal{F}_t) \to 1$ on A. Consequently,

$$Y_n := I_A \pi_t(G) - I_A \varepsilon/2 + I_A V_n - I_A I_{B_n}[\pi_t(G) - \varepsilon/2 + V_n - G]$$

tends to GI_A in probability: indeed,

$$P(Y_n \neq GI_A) \leq P(B_n^C \cap A) = E1_A P(B_n^C|\mathcal{F}_t) \to 0, \ n \to \infty.$$

Clearly, $Y_n \in I_A(Q - L^0_+)$. Necessarily, $GI_A \in \overline{1_A(Q - L^0_+)}$, where closure is taken in the topology of stochastic convergence. But it is well-known (see arguments of Kabanov and Stricker (2001)) that under (NA) the set $1_A(Q - L^0_+)$ is closed in probability. This means that for some $\tilde{V} \in \mathcal{V}_t$ and $l \in L^0_+$ we have

$$GI_A = 1_A \pi_t(G) - 1_A \varepsilon/2 + 1_A \tilde{V} - 1_A l.$$

For $k = t+1, \ldots, T$ calling $\hat{\phi}_k$ a super replication strategy for $\pi_k(G)$,

$$1_A(\pi_t(G) - \varepsilon/2) + 1_{A^c}\pi_t(G) + 1_A \tilde{V} + 1_{A^c} \sum_{k=t+1}^{T} \hat{\phi}_k \Delta S_{k+1} \geq G,$$

which is contradicts (19), so the statement is proved. □

We also recall the following Lemma from Carassus and Rásonyi (2005) :

Lemma 5.4 *Suppose that U_n, $n \in \mathbb{N}$ satisfy Assumption 2.3 as well as $U_n(0) = 0$, $U_n'(0) = 1$, for all $n \in \mathbb{N}$. Then*

$$\forall y < 0 \quad U_n(y) \to -\infty, \ n \to \infty, \quad \forall y \geq 0 \quad U_n(y) \to 0, \ n \to \infty.$$

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