Graph colorings with local constraints — A survey *

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Abstract

We survey the literature on those variants of the *chromatic number* problem where not only a proper coloring has to be found (i.e., adjacent vertices must not receive the same color) but some further local restrictions are imposed on the color assignment. Mostly, the *list colorings* and the *precoloring extensions* are considered.

In one of the most general formulations, a graph $G = (V, E)$, sets $L(v)$ of admissible colors, and natural numbers c_v for the vertices $v \in V$ are given, and the question is whether there can be chosen a subset $C(v) \subseteq L(v)$ of cardinality c_v for each vertex in such a way that the sets $C(v), C(v')$ are disjoint for each pair v, v' of adjacent vertices. The particular case of constant $|L(v)|$ with $c_v = 1$ for all $v \in V$ leads to the concept of *choice number*, a graph parameter showing unexpectedly different behavior compared to the chromatic number, despite these two invariants have nearly the same value for almost all graphs.

To illustrate typical techniques, some of the proofs are sketched.

Keywords: graph coloring, list coloring, choice number, precoloring extension, complexity of algorithms, chromatic number

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Contents

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0 Introduction

The key concept of this survey, *list coloring*, was introduced in the second half of the 1970s, in two papers, by Vizing [190] and independently by Erdős, Rubin and Taylor [62]. Despite the subject offers a large number of challenging problems, some of which appeared already in [62], the vertex list colorings remained almost forgotten for about a decade. The field started to flourish around 1990, and has attracted an increasing attention since then. Most of the early questions have been answered, and new directions have been initiated. But one of the innocent-looking problems raised in [62] (Problem 1.5 below) is still open, and in the particular cases for which affirmative answers have been proved, we are still rather far from a general solution.

The systematic study of precoloring extensions was initiated about a decade after [62], in the paper by Biró, Hujter and Tuza [18]. Some of its particular cases (mostly in connection with edge colorings) appeared earlier in the works of Burr [40], Marcotte and Seymour [145], and, using a different terminology, in several papers on Latin squares.

In this paper we summarize what is known so far on these problems and in their 'close neighborhood.' Surveying this part of the literature, not only the strongest results but also much of the history is presented. Some typical techniques are illustrated by sketches of proofs. Several open problems are mentioned, too.

We have to mention at this point that the class of hypergraphs seems to offer a big unexplored area with many interesting results to be discovered. And, in this context as well, the intensively studied symmetric structures (finite geometries, Steiner systems, balanced incomplete block designs) may deserve more attention.

There are at least two previous works to be cited for general reference on list colorings. The paper of Alon [4] surveys the early results, presents some of the important methods, and also contains several new theorems. Moreover, many aspects of list colorings, with lots of interesting historical remarks and informative comments, are discussed in various subsections of the excellent book by Jensen and Toft [111].

Applications. Before giving the formal definitions, let us mention that both List Coloring and Precoloring Extension are well motivated, providing natural interpretations for various kinds of scheduling problems; see, e.g., [18, 19, 22]. As a matter of fact, the starting point of the investigations on precoloring extension was the observation that, on interval graphs, it provides an equivalent formulation of a practical problem where flights have to be assigned to a given number of airplanes according to the schedule of a timetable, under the additional condition that the fixed schedule of maintenances (prescribed for each airplane) must not be changed. Further applications include issues in VLSI theory. The problem of T-Colorings has important practical motivation as well, from the area of frequency assignments to avoid interferences; see [89, 174] and the surveys [154, 155]. Precoloring extension also has some consequences on the non-approximability of some scheduling problems [22]. Moreover, edge colorings of complete bipartite (and also of complete) graphs have equivalent interpretations in terms of Latin squares and rectangles. The extendability of partial Latin squares has been studied extensively; we refer to the survey [10] and the more recent paper [11] for references in this part of the literature.

From the theoretical point of view, Vizing introduced list colorings with the intention to study total colorings, while Erdős, Rubin and Taylor took their motivation from Dinitz's conjecture on $n \times n$ matrices. Last but not least, the idea of extending a partial coloring to a larger one is a natural approach in various contexts where graph colorings are constructed sequentially.

Related problems. At the end of this informal introduction, let us say a few words also about three topics that will *not* be considered here, despite they might have fitted nicely in the context. First, we shall not deal with problems in which some forbidding condition (e.g., the exclusion of 'being monochromatic') is extended from adjacent vertices to vertex pairs at larger distance apart. These 'distance colorings' lead to interesting questions and results, but usually may be viewed as colorings on the kth powers of graphs, and so they are less 'restricted' than the concepts discussed here. Second, in a more general setting, the '$\mathcal{P}$-chromatic number' of a graph can be defined with respect to any hereditary property $\mathcal{P}$. This concept is discussed in detail in the paper [33], therefore we shall only mention a couple of related references at some points. Last but not least, we do not consider here 'rankings,' i.e., vertex (edge) colorings with positive integers in such a way that each monochromatic pair of vertices (edges) is completely separated by the vertices (edges) of greater colors. A large part of the literature on rankings can be traced back from the relatively recent papers [125] and [21]. The unpublished manuscript [184] surveys many problems; we hope to polish this preliminary version for publication reasonably soon.

0.1 Standard definitions

A graph (meant to be undirected, without loops and multiple edges) or multigraph (undirected, without loops) will usually be written in the form $G = (V, E)$, where $V = V(G)$ and $E = E(G)$ denote the set of vertices and edges, respectively. The *complement* of G will be denoted by $\overline{G}$, the *degree* of vertex v by $d(v)$ or $d_H(v)$ when the particular graph H in which it is considered has to be emphasized, and the *maximum degree* of G by $\Delta(G)$ or Δ. The cardinality $|V|$ of the vertex set is called the *order* of G, and usually will be denoted by n. The parameters $\alpha(G)$ and $\omega(G)$ denote the *independence* or *stability number* and the *clique number*, respectively (i.e., the largest cardinality of a subset $Y \subseteq V$ consisting of mutually non-adjacent resp. adjacent vertices). Standard notation is applied for particular types of graphs, too, including K_n

(complete graph with n vertices), $K_{p,q}$ (complete bipartite graph with vertex classes of respective cardinalities p and q), P_n (path of length $n-1$), C_n (cycle of length n), S_n (star of degree $n-1$). Terminology not defined here for particular classes of graphs and basic concepts can be found e.g. in [15, 29, 72, 91, 142].

A *proper vertex / edge / total coloring* is a mapping φ from the set V / E / $V \cup E$ into the set $\mathbb{N}$ of natural numbers, such that the first / the second / all the three conditions below are satisfied:

- $\varphi(v) \neq \varphi(v')$ for all vertex pairs $v, v' \in V$ with $vv' \in E$,
- $\varphi(e) \neq \varphi(e')$ for each pair $e, e' \in E$ of edges sharing a vertex,
- $\varphi(v) \neq \varphi(e)$ for all $v \in V$ and all $e \in E$ with $v \in e$.

Throughout the paper, the expressions 'coloring' and 'proper coloring' will be used as synonyms, except in the few paragraphs where the 'T-colorings' are considered (see the definition in Subsection 0.3). We shall mostly deal with vertex colorings; the only exceptions are some complexity issues (in Section 4) and the material presented in Section 3.3.

0.2 Notation for vertex colorings

Assuming that the vertex set is $V = \{v_1, \ldots, v_n\}$, L_i will denote the list (= set of admissible colors) associated with v_i. For the union of the lists, we use the notation

$$\mathbb{L} := L_1 \cup \cdots \cup L_n .$$

We also denote

$$\mathcal{L} := (L_1, \ldots, L_n) ,$$

the (ordered) n-tuple of lists. A mapping $\varphi : V \to \mathbb{L}$ is a (vertex) *list coloring*, or an *$\mathcal{L}$-coloring*, if φ is a proper coloring and $\varphi(v_i) \in L_i$ holds for all $1 \leq i \leq n$. (In some papers, the set of *forbidden* colors is given instead of the admissible ones. Those sets may be viewed as complements of the L_i with respect to $\mathbb{L}$.)

If $|L_i| = k$ for all i, then $\mathcal{L}$ is termed a *k-assignment.* The *choice number* of G (also called the *list chromatic number* in the literature), denoted $\chi_\ell(G)$, is the smallest k such that every k-assignment $\mathcal{L}$ admits a list coloring. For $\chi_\ell(G) \leq k$, G is said to be *k-choosable.* Since the identical lists (defining $L_i := \{1, \ldots, k\}$ for all i) form a particular k-assignment, it follows by definition that the *chromatic number* $\chi(G)$ of G does not exceed $\chi_\ell(G)$.

The concept of *precoloring extension* lies between k-colorability and k-choosability. In this problem, a vertex subset $W \subset V$ of the graph $G = (V, E)$ is *precolored* with

$$\varphi_W : W \to \{1, \ldots, k\}$$

for some $k \in \mathbb{N}$, where the mapping φ_W is *not* required to be onto (and, in particular, $W = \emptyset$ is also allowed), and the question is whether G admits a proper k-coloring that extends φ_W. That is, a color should be assigned to each *precolorless* vertex $v_i \in V \setminus W$ from the list $L_i := \{1, \ldots, k\}$ (identical lists for the entire $V \setminus W$) while the colors $L_j := \{\varphi_W(v_j)\}$ of the *precolored* vertices $v_j \in W$ are unchangeable. The parameter k is termed the *color bound.*

Finally, the *coloring number* of G, denoted $\mathrm{col}(G)$, is defined as the largest integer k such that G has a subgraph of minimum degree $k-1$. Equivalently, $\mathrm{col}(G)$ is the smallest k such that G is '$(k-1)$-degenerate.' As a trivial first remark, let us note that if v_n has more colors in its list than the number of its neighbors, then G is list colorable if and only if so is $G - v_n$. In this way, the inequalities

$$\chi(G) \leq \chi_\ell(G) \leq \mathrm{col}(G) \leq \Delta(G) + 1$$

are valid for every graph G.

0.3 Some variations

Beside the concepts introduced above, at some points we shall mention results on the following variants, too.

(f, g)-choosability. A more general setting for k-choosability is as follows. Let f and g be two functions from the same domain V into $\mathbb{N}$, with $f(v_i) \geq g(v_i)$ for all $1 \leq i \leq n$. The graph G is said to be (vertex-) (f, g)-*choosable* if, for every list assignment $\mathcal{L}$ with $|L_i| = f(v_i)$ for all i, there can be chosen subsets $S_i \subseteq L_i$ of cardinality $|S_i| = g(v_i)$, such that $S_i \cap S_j = \emptyset$ holds for every edge $v_i v_j \in E$. The constant functions are of particular interest; the case $g \equiv 1$ is termed *f-choosable*, while $f \equiv k$ and $g \equiv \ell$ with $k, \ell \in \mathbb{N}$ fixed will be referred to as *(k, ℓ)-choosable.* These concepts extend to edge and total colorings in the natural way.

(p, q, r)-choosability. This type of list colorings is obtained from the previous one by taking constant functions $f \equiv p$ and $g \equiv q$, and assuming that $|L_i \cup L_j| \geq p + r$ whenever v_i and v_j are adjacent. To exclude trivial uncolorability, it is assumed that $p \geq q$ and $p + r \geq 2q$.

List T-colorings. Given a set $T \subset \mathbb{N} \cup \{0\}$, a (vertex) *$T$-coloring* of $G = (V, E)$ is a mapping $\varphi : V \to \mathbb{N}$ such that $|\varphi(v_i) - \varphi(v_j)| \notin T$ holds for all edges $v_i v_j \in E$. List T-colorings are defined in the natural way, choosing each color $\varphi(v_i)$ from the corresponding list L_i. The *T-choice number*, i.e., the smallest k for which every k-assignment of G has a list T-coloring, will be denoted by $\chi_{\ell|T}(G)$. Note that a (list) T-coloring is required to be a proper coloring if and only if $0 \in T$; in fact, a list coloring is a list T-coloring with $T = \{0\}$, and $\chi_{\ell|\{0\}} = \chi_\ell$ holds.

0.4 Small uncolorable graphs

We close this introduction with some simple examples admitting no list coloring, to illustrate the above definitions.

Example 0.1. *The complete bipartite graph $K_{2,4}$ with the lists $\{1,2\}$ and $\{3,4\}$ in the first vertex class and $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$ in the second class admits no* *list coloring, hence it is not 2-choosable. Similarly, $K_{3,3}$ with the lists $\{1,2\}$, $\{1,3\}$, and $\{2,3\}$ in each vertex class has no list coloring, therefore it is not 2-choosable either. On the other hand, it is easy to show that both graphs are 3-choosable, thus $\chi_\ell(K_{2,4}) = \chi_\ell(K_{3,3}) = 3$ holds.*

Example 0.2. *One of the simplest non-3-choosable, planar, K_4-free graphs is obtained from $K_{2,18}$ by inserting a matching of nine edges in the 18-element vertex class. Denote these edges by e_{ij}, where $1 \le i \le 3$ and $4 \le j \le 6$. Assign the lists $\{1,2,3\}$ and $\{4,5,6\}$ to the vertices in the 2-element class; and the list $\{i,j,7\}$ to both vertices of each matching edge e_{ij}. This 3-assignment admits no list coloring.*

Example 0.3. *A non-3-choosable bipartite graph with transparent structure is $K_{7,7}$, e.g. with the lists $\{1,2,3\}$, $\{1,4,5\}$, $\{1,6,7\}$, $\{2,4,7\}$, $\{2,5,6\}$, $\{3,4,6\}$, $\{3,5,7\}$ in each vertex class. These lists correspond to the seven lines of the Fano plane, where the colors are viewed as points. It is well known (and easy to see) that if a set T of at most three points meets all lines of the plane, then T itself is a line. Thus, in any selection of colors from the above lists, either at least four of the seven colors occur in each vertex class, or in one class the three colors of an entire line are selected (and this line is a list in the other class, too). In either case, some color is selected in both classes, implying that no list coloring exists because the corresponding two vertices are adjacent.*

Example 0.4. *Consider the list T-coloring problem on $K_{3,3}$ with lists $\{1,2\}$, $\{1,3\}$, $\{2,3\}$ in one vertex class and $\{1,3\}$, $\{1,4\}$, $\{3,4\}$ in the other class, where $T = \{2\}$. Though $0 \notin T$, no feasible coloring exists. (Compared to Example 0.1, the lists are now 'shifted' by 2 (mod 4).) The graph remains uncolorable even if we remove the two edges $(\{1,3\},\{1,3\})$ and $(\{2,3\},\{1,4\})$.*

1 General results

In this section we review some of the most general facts, walking around the subject from several different sides.

1.1 Equivalent formulations

Next, we present two types of reductions, taken from [18] and [190], respectively. The first one shows in two steps that the three problems of list coloring, precoloring extension, and chromatic number are quickly reducible to each other. (In one direction it is obvious that, in general, list coloring is hardest and chromatic number is the most particular case, with all lists identical and having no precolored vertices.) The second construction will establish a relationship between list colorability and independence number.

List colorings vs. precoloring extension. Assuming that a graph $G = (V,E)$ with a list assignment $\mathcal{L}$ is given, and that the union $\mathbb{L}$ of the lists is the interval $\{1,\dots,k\}$ without any gaps, take k new vertices $u_1,\dots,u_k$ and join u_i with v_j if and only if $i \notin L_j$ ($1 \le i \le k$, $1 \le j \le n$). Then, forgetting about the list assignment, precolor the vertex u_i with color i, for all $i = 1,\dots,k$. This precoloring of the larger graph is extendable with color bound k (i.e., without taking any new colors) if and only if G is list colorable.

Precoloring extension vs. chromatic number. Let the graph $G = (V,E)$ with precolored set W and color bound k be given. Assuming that $W_i \subseteq W$ is the (possibly empty) set of vertices of color i for $1 \le i \le k$, replace W_i by a new vertex u_i (joining $v_j \in V \setminus W$ to u_i if and only if v_j had at least one neighbor in W_i), and make the new vertices u_i mutually adjacent, creating a complete subgraph of order k. The modified graph has chromatic number k if and only if the precoloring of G is extendable with color bound k.

List colorings vs. independence number. Given the graph $G = (V,E)$ with a list assignment $\mathcal{L}$, construct the graph $G\square\mathcal{L}$ with vertex set

$$V(G\square\mathcal{L}) := \{(i,j) \mid v_i \in V, j \in L_i\}$$

and join two of its vertices $(i',j'),(i'',j'')$ if and only if they belong to the same vertex (i.e., $1 \le i' = i'' \le n$) or to the same color at adjacent vertices ($j' = j'' \in L_{i'} \cap L_{i''}$ and $v_{i'}v_{i''} \in E$).

Theorem 1.1. *(Vizing [190]) The graph $G = (V,E)$ with lists $\mathcal{L}$ admits a list coloring if and only if $\alpha(G\square\mathcal{L}) = n$.*

As a matter of fact, slightly more is true; namely, there is a bijection between the admissible list colorings and the independent sets of cardinality n, as the vertex set of $G\square\mathcal{L}$ is partitioned into the n cliques induced by the sets $\{(i,j) \mid j \in L_i\}$, $1 \le i \le n$. Note further that if all lists are identical, then the above construction results in the known equivalent definition of the chromatic number, stating that a graph $G = (V,E)$ is k-colorable if and only if the 'Cartesian product' (also called 'box product') of G and K_k has independence number $|V|$.

1.2 Complete bipartite graphs and the construction of Hajós

Next, we consider complete bipartite — and more generally, complete multipartite — graphs, present estimates on their choice numbers, and show how they can be taken as building blocks to construct all non-k-choosable graphs.

We have already seen (cf. Examples 0.1 and 0.3) that some bipartite graphs are not 2-choosable. As a matter of fact, the choice number of $K_{n,n}$ tends to infinity with n, and its growth can be described fairly accurately along the following observations of [62].

Denote by m_r the minimum number of edges in an r-uniform 3-chromatic hypergraph $\mathcal{H}_r$ (i.e., $|H| = r$ for all $H \in \mathcal{H}_r$, and in every vertex partition of $\mathcal{H}_r$ into two parts, at least one part contains some $H \in \mathcal{H}_r$). View the vertices of $\mathcal{H}_r$ as colors, and assign the edges of $\mathcal{H}_r$ to the vertices in each vertex class of $K_{n,n}$, for any $n \geq m_r$, as lists. If there were a list coloring (in which no color appears in both classes of $K_{n,n}$), it would yield a 2-partition of $\mathcal{H}_r$ with no part containing any $H \in \mathcal{H}_r$; thus, $\chi_\ell(K_{n,n}) > r$. On the other hand, if $2n < m_r$, then the lists in every list assignment on the vertices of $K_{n,n}$ form some 2-chromatic hypergraph $\mathcal{H}$, and from a proper 2-partition $A \cup B$ of $\mathcal{H}$, we can choose a color from A for the vertices in one class of $K_{n,n}$ and a color from B for the other class. Thus, the smallest $n = n_r$ for which $K_{n,n}$ is not r-choosable satisfies the inequalities

$$\tfrac{1}{2}\, m_r < n_r \leq m_r\,.$$

It is known (see [14, 59, 61]) that the growth of m_r is between $r^{1/3-\varepsilon} \cdot 2^r$ and $r^2 \cdot 2^r$, therefore we obtain

Theorem 1.2. *(Erdős, Rubin, Taylor [62]) As $n \to \infty$, $\chi_\ell(K_{n,n}) = \log_2 n + o(\log n)$.*

Unequal vertex classes. The exact determination of $\chi_\ell(K_{p,q})$ seems to be hopeless. Already the simplest particular case of characterizing all pairs p, q with 3-choosable $K_{p,q}$, $p \leq q$, required much effort. A complete description can be obtained by combining the works of Mahadev, Roberts and Santhanakrishnan [144] ($p = 3$, $q \leq 26$, and $p = 4$, $q \leq 18$), Füredi, Shende and Tesman [169] ($p = 5$, $q \leq 12$), and O-Donnel [151] ($p = 6$, $q \leq 10$).

A related problem is to determine the smallest n_k, $k \in \mathbb{N}$, for which there exists a non-k-choosable $K_{p,q}$ with $p + q = n_k$. By the observations above, $n_k \leq 2m_k$ holds, and for small values the bound is tight: $n_2 = 6$ and $n_3 = 14$ (the latter proved in [90], cf. Example 0.3). Hanson, MacGillivray and Toft also give the recursive estimate $n_k \leq k\, n_{k-2} + 2^k$ which, for k even, provides a better upper bound than the known ones for $2m_k$ (e.g., $n_4 \leq 40$ is obtained, while the currently best upper bound on m_4 is 23). Nevertheless, as sufficiently strong lower bounds on m_k are not available, the equality of n_k and $2m_k$ has not been ruled out for any value of k so far.

For p much bigger than q, Hoffman and Johnson [98] determine the choice number, proving that

$$\chi_\ell(K_{p,q}) = q \qquad \text{for} \qquad (q-1)^{q-1} - (q-2)^{q-1} \leq p < q^q$$

and that for p smaller, χ_ℓ is smaller than q. (For $p \geq q^q$, the choice number is easily seen to equal $q + 1$.) They also study which list assignments are uncolorable when p is at the two ends of the above interval. It is not known, however, for which values of p the smaller 'jumps' in χ_ℓ occur as it grows from $\log_2 q$ to $q - 1$ (for q fixed). For instance, if $q = 6$, we obtain $\chi_\ell(K_{p,6}) = 5$ for all $125 \leq p \leq 5^5 - 4^5 - 1 = 2100$ (the construction for $p = 125$ is due to Eaton [54]), but it is not clear whether this is indeed the entire range of p for $\chi_\ell = 5$.

Complete multipartite graphs. More generally, denote by K_{r*t} the complete r-partite graph with t vertices in each of its r classes. Alon [3] proves that there exist positive constants c_1, c_2 such that

$$c_1\, r \log t \leq \chi_\ell(K_{r*t}) \leq c_2\, r \log t \tag{1}$$

holds for every r and t.

Generating all non-k-choosable graphs. In his well known paper [88], Hajós describes three elementary operations by the repeated application of which all graphs of chromatic number greater than k can be obtained from the complete graph K_{k+1}. Gravier has proved that a similar generating procedure can be applied for graphs whose choice number exceeds k.

Consider the following three types of operations.

1. Add a new vertex or a new edge.
2. Let G_1, G_2 be vertex-disjoint, and $x_i y_i$ an edge in G_i, $i = 1, 2$. Identify x_1 and x_2, join y_1 with y_2 by a new edge, and delete the edges $x_1 y_1$ and $x_2 y_2$.
3. If G has an uncolorable list assignment $\mathcal{L}$ such that $|L_i| \geq k$ for all $1 \leq i \leq n$ and two nonadjacent vertices $v_{i'}, v_{i''}$ have the same list in $\mathcal{L}$, then identify $v_{i'}$ with $v_{i''}$.

Theorem 1.3. *(Gravier [74]) Every non-k-choosable graph can be generated by the above three operations from any one non-k-choosable complete bipartite graph.*

In this way, the role of complete graphs is taken by the complete bipartite ones when χ is replaced by χ_ℓ. It is interesting to note that, though there is an increasing number of (inclusionwise) minimal complete bipartite non-k-choosable graphs as k gets large, all of them are equivalent from the generative point of view.

1.3 Typical behavior of the choice number

In this subsection we present asymptotic results on the choice number of random graphs and random bipartite graphs.

On one hand, putting $r := \chi(G)$ and $t := n$, the inequality (1) implies

$$\chi_\ell(G) \leq c\, \chi(G) \log n$$

for every graph of order n, for some absolute constant c. On the other hand, the complete bipartite graphs $K_{n,n}$ already show that this bound is tight (apart from the actual value of c), and, in particular, the choice number is not bounded by any function of the chromatic number. In this setting it may be unexpected that, nevertheless, χ and χ_ℓ have nearly the same value for *almost all* graphs.

Random graphs. Let p be a real number, $0 < p < 1$. Denote by $G_{n,p}$ the random graph on n vertices, in which each unordered vertex pair $v_i v_j$ is chosen to be an edge with probability p, and these choices are made totally independently of each other. The following result for $p = 1/2$ is due to J. Kahn (its proof appeared in [4]); the general case has been proved by Tuza and Voigt [182]. (The weaker upper bound of $o(n)$, conjectured in [62], was first proved by Alon [3].)

Theorem 1.4. *For every fixed edge probability p,*

$$\chi_\ell(G_{n,p}) = (1 + o(1)) \cdot \chi(G_{n,p})$$

with probability $1 - o(1)$ as $n \to \infty$.

An important result of Bollobás [25] states that

$$\chi(G_{n,p}) = \left(\frac{1}{2} + o(1)\right) \left(\log \frac{1}{1-p}\right) \frac{n}{\log n},$$

i.e., the expected value of the chromatic number asymptotically equals the order n divided by the expectation of the independence number. As regards the choice number, one can prove that there exists a slowly decreasing sequence $\epsilon_n \to 0$ (the appropriate speed of convergence can be read out from numerical estimates of [25]) for which the following procedure successfully finds a list coloring for any k-assignment with $k = (1 + \epsilon_n) \cdot \chi(G_{n,p})$. As long as there exists an independent set S of at least $(1 - \epsilon'_n) \cdot \alpha(G_{n,p})$ currently uncolored vertices and a color i appearing in the lists of all vertices in S, assign i to the entire S and remove i from all the other lists. On the other hand, if such a large uncolored S does not exist anymore, then, for every subset Y of the currently uncolored vertices, the union of the modified lists belonging to Y contains at least $|Y|$ colors, thus the remaining lists have distinct representatives by the Kőnig–Hall theorem.

It remains an open problem to settle whether $\chi(G_{n,p})$ and $\chi_\ell(G_{n,p})$ have the same asymptotic behavior for every 'reasonable' edge probability function $p = p(n)$. Neither is it known how strongly χ_ℓ is concentrated, and whether $\chi_\ell > \chi$ holds with probability $1 - o(1)$.

Random bipartite graphs. Erdős, Rubin and Taylor [62] investigated the random bipartite graph $B_{n,p}$ with $m = n/2$ vertices in each class and with edge probability $p = 1/2$ They proved the logarithmic growth of

$$\frac{\log m}{\log 6} < \chi_\ell(B_{n,p}) < \frac{3 \log m}{\log 6}$$

with probability $1 - o(1)$ as $m \to \infty$.

1.4 Unions of graphs and the (am, bm)-conjecture

In this section we deal with some problems and results related to (k, ℓ)-choosability. Perhaps the most challenging open question of this kind is the following one, being unsolved for already almost two decades.

Problem 1.5. *(Erdős, Rubin, Taylor [62]) If G is (k, ℓ)-choosable, does it follow that G is $(km, \ell m)$-choosable for every $m \in \mathbb{N}$?*

It is widely believed that the answer is affirmative (justifying the word 'conjecture' in the title of this subsection), and almost all known proofs showing that a certain graph is $(k, 1)$-chosable can be extended with little effort to verify (km, m)-choosability. Nevertheless, $(k, \ell) = (2, 1)$ is the only case for which the implication formulated in Problem 1.5 has been proved for all m and for all graphs G satisfying the supposition (i.e., for all 2-choosable graphs). This result, published in [186], can be extended to obtain a reduction method as follows.

Theorem 1.6. *(Tuza, Voigt [187]) Let $\mathcal{L}$ be a k-assignment on $G = (V, E)$, and suppose that $X \subset V$ is a vertex set such that the edges incident to X form a 2-choosable graph. Then, there can be chosen a color $\varphi(v_i) \in L_i$ for each $v_i \in X$, in such a way that*

$$\left| L_j \cap \{\varphi(v_i) \,|\, v_i \in X, v_i v_j \in E\} \right| \leq 1$$

holds for every $v_j \in V \setminus X$.

If a set $X \subset V$ with the above property exists in G and, in addition, the induced subgraph $G - X$ can be proven to be $(k-1)$-choosable, then the k-choosability of G follows as well. The (km, m)-choosability of G can be deduced in a similar way; for instance, the $(3m, m)$-choosability of the Petersen graph is obtained for every $m \in \mathbb{N}$.

Graph union. One of the interesting consequences of an affirmative answer to Problem 1.5 (if it holds true indeed) would be that the choice number is a submultiplicative function with respect to graph union. For the time being, however, this can only be formulated as yet another intriguing open problem.

Conjecture 1.7. *For any two graphs G_1 and G_2 on the same vertex set,*

$$\chi_\ell(G_1 \cup G_2) \leq \chi_\ell(G_1)\, \chi_\ell(G_2). \tag{2}$$

To see that the implication of Problem 1.5 would indeed imply (2), assume $\chi_\ell(G_i) = k_i$ for $i = 1, 2$. Starting with any $(k_1 k_2)$-assignment of $G_1 \cup G_2$, choose k_2-element color sets $S_i \subseteq L_i$ such that S_i and S_j are disjoint whenever $v_i v_j \in E(G_1)$ — on applying that the k_1-choosability of G_1 implies its $(k_1 k_2, k_2)$-choosability as well — and then find a list coloring of the k_2-choosable graph G_2 in the list assignment $(S_1, \ldots, S_n)$. (More generally, inserting the edges of a (b, c)-choosable graph into an (a, b)-choosable graph, we obtain an (a, c)-choosable one [62].) By the results cited above, the inequality (2) holds (with equality) if at least one of the two G_i is 2-choosable.

Jensen and Toft [111] remark that so far (2) is not confirmed even for the following rather simple particular case. Suppose that G is bipartite, and substitute two nonadjacent vertices for each vertex of G. (Each edge of G becomes then an induced C_4.) It is easily seen that the new graph G' can be written in the form $G_1 \cup G_2$, where $G_1 \simeq G_2 \simeq 2G$; it is not known, however, whether $\chi_\ell(G') \leq (\chi_\ell(G))^2$.

Let us mention here a further problem, that deals with the union of *three* graphs.

Conjecture 1.8. (Voigt [192]) *Let $G = (V, E)$ be a graph with $V = V_1 \cup V_2 \cup V_3$ where V_1, V_2, V_3 are mutually disjoint independent sets, and suppose that the subgraph induced by $V_i \cup V_j$ is 2-choosable for all $1 \le i < j \le 3$. Then G is 3-choosable.*

Recently, Voigt proved in [194] that those graphs are 4-choosable, and more generally, $(4m, m)$-choosable for all $m \in \mathbb{N}$.

Though the inequality (2) has not yet been proved, an upper bound on the choice number of the union of two graphs follows from a result of Alon (Theorem 2.5, to be discussed later).

Theorem 1.9. *(Alon) There exists a function $h\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $\chi_\ell(G_1 \cup G_2) \le h(\chi_\ell(G_1), \chi_\ell(G_2))$ holds for any two graphs G_1, G_2.*

The superexponential upper bound read out from Theorem 2.5 is the best known general one, hence being very far from quadratic (as expected).

1.5 Graphs and their complements

The well known theorem of Nordhaus and Gaddum states that

$$\chi(G) + \chi(\overline{G}) \le n + 1$$

holds for every graph G on n vertices. As shown in [62], this inequality can be strengthened to a far extent.

Theorem 1.10. *(Erdős, Rubin, Taylor [62]) Every graph G of order n satisfies*

$$\chi_\ell(G) + \chi_\ell(\overline{G}) \le col(G) + col(\overline{G}) \le n + 1\,.$$

For a short proof, denote by d_i^- the number of vertices v_j with $j < i$ that are adjacent to v_i, and by d_i^+ the number of those v_j with $j > i$ which are *nonadjacent* to v_i. Assuming that the vertices are labelled in a decreasing order of degree, we see

$$d_i^- + d_j^+ \le n - 1 \qquad \forall\, i \ge j\,.$$

(The inequality remains valid even if we replace d_i^- and d_j^+ by the degrees $d_G(v_i)$ and $d_{\overline{G}}(v_j)$ in the entire G and $\overline{G}$, respectively.) Moreover, since $d_i^- \le i-1$ and $d_i^+ \le n-i$ obviously hold, we also have

$$d_i^- + d_j^+ \le i - 1 + n - j \le n - 1 \qquad \forall\, i \le j\,.$$

The combination of the above inequalities yields

$$\chi_\ell(G) + \chi_\ell(\overline{G}) \le \mathrm{col}(G) + \mathrm{col}(\overline{G}) \le \max_i (d_i^- + 1) + \max_j (d_j^+ + 1) \le n + 1\,,$$

proving the assertion.

The graphs attaining equality in the Nordhaus–Gaddum theorem have been described (cf. [15]). On the other hand, the following problem seems to be unsolved, as well as its analogue for the coloring number.

Problem 1.11. *Characterize the structure of graphs $G = (V, E)$ such that*

$$\chi_\ell(G) + \chi_\ell(\overline{G}) = |V| + 1\,.$$

To see *how small* the sum $\chi_\ell(G) + \chi_\ell(\overline{G})$ can be, consider the complete r-partite graph $G := K_{r*t}$ of order $n = rt$, with $r := \sqrt{n/\log n}$ vertex classes and $t := \sqrt{n \log n} = n/r$ vertices in each class. Applying the upper bound of Inequality (1), we obtain that $\chi_\ell(G) = O(\sqrt{n \log n})$; moreover, its complement $\overline{G}$ is just rK_t, so that $\chi_\ell(G) + \chi_\ell(\overline{G}) = O(\sqrt{n \log n})$. (This construction in [3] answered a question of [62] in the negative.)

It is an open problem whether the factor $\sqrt{\log n}$ is necessary in the formula, or perhaps $\chi_\ell(G_n) + \chi_\ell(\overline{G}_n) \le c\sqrt{n}$ holds for an infinite sequence of graphs G_n of order n.

2 Vertex degrees

In this section we discuss three main issues. The first one is to investigate the possible extensions of Brooks's theorem for various types of choosability, i.e., to obtain sufficient conditions in terms of vertex degrees for choosing colors or color sets from the lists. The second one is a lower bound on $\chi_\ell(G)$ as a function of the average degree, a property in which the choice number significantly differs from the chromatic number. The third and fourth subsections are devoted to an algebraic approach invented by Alon and Tarsi, that leads to sufficient conditions for choosability, in terms of the existence of certain orientations on the edges.

The bounds on *edge* colorings are also strongly related to vertex degrees, but we shall discuss them only later, in Section 3.3.

2.1 The theorems of Brooks and Gallai

The inequality $\chi_\ell(G) \le \mathrm{col}(G) \le \Delta(G) + 1$ yields an obvious upper bound on the choice number. Certainly, the bound is tight, and one nice class attaining equality is that of the chordal graphs. In fact, arranging the vertices of a chordal graph in reversed simplicial order $v_1, \ldots, v_n$ (i.e., where for each $i \le n$, the neighbors v_j of v_i with $j < i$ are mutually adjacent), gives not only a simple coloring algorithm, but also demonstrates that the bounds obtained are best possible. In this way, one can handle many situations, including (f, g)-choosability and T-choosability as well (see e.g. [174, 175, 185]). For instance, it is easy to show that denoting by ω_i the largest number of vertices in a clique containing v_i, every chordal graph is (f, g)-choosable for $f(v_i) = m\,\omega_i$ and $g \equiv m$, for all $m \in \mathbb{N}$.

Similarly to the classic theorem of Brooks [38], the previous upper bounds on the necessary length of lists are hardly ever tight, and lists of lengths not exceeding the vertex degrees suffice in most graphs. The first result of this kind is due to Vizing [190] who proved that a connected graph of maximum degree Δ is Δ-choosable unless it is $K_{\Delta+1}$, or $\Delta = 2$ and the graph is an odd cycle. Erdős, Rubin and Taylor [62] and Borodin [30] strengthened this assertion, proving list colorability with lists of lengths $d(v_i)$ for every vertex v_i, provided that at least one 2-connected block of the connected graph is not a clique or an odd cycle. Tuza and Voigt [185] showed further that, under the same structural condition, color sets of cardinality m can be chosen whenever $|L_i| = m\,d(v_i)$ for every v_i. We summarize these results in the following assertion.

Theorem 2.1. *Let $m \in \mathbb{N}$, and let $G = (V, E)$ be a connected graph. Suppose that $\mathcal{L}$ is a list assignment where $|L_i| \geq m\,d(v_i)$ for each $v_i \in V$. If*

(i) $|L_i| > m\,d(v_i)$ for some v_i, or

(ii) G contains a block which is neither a complete graph nor an induced odd cycle,

then G admits a choice of an m-element $C_i \subseteq L_i$ for each i, such that $C_i \cap C_j = \emptyset$ for all $v_iv_j \in E$.

Further generalizations are known for list T-colorings (Waller [196], also making a distinction for the cases where T is an arithmetic progression containing 0) and colorings with respect to additive and hereditary graph properties (Borowiecki et al. [35, 34]). The previous theorem does not hold true for infinite graphs, however, as shown by the following class of examples. Take the countable star S^* with center v_0 and leaves $v_1, v_2, \ldots$, with the list assignment $L_0 = \mathbb{N}$ and $L_i = \{i\}$ for all $i \in \mathbb{N}$, and join v_0 with a vertex of a finite 2-connected graph G which is neither a complete graph nor an odd cycle. If the lists on G are larger than $|V(G)|$, then the conditions of the theorem are satisfied in the graph composed from S^* and G, but no list coloring exists since already S^* is uncolorable.

Critical graphs. A closely related classic theorem due to Gallai [69] deals with the structure of subgraphs induced by the set of vertices of minimum degree in a color-critical graph. To generalize this result, call a graph $G = (V, E)$ *critical* with respect to a color assignment $\mathcal{L}$ if it has no list coloring, but each of its proper induced subgraphs does have one. Clearly, $|L_i| \leq d(v_i)$ holds for every vertex v_i if G is critical. Call v_i *small* if its degree equals $|L_i|$.

Theorem 2.2. *(Kostochka, Stiebitz, Wirth [132]; Thomassen [178]) If the graph G is critical with respect to the list assignment $\mathcal{L}$, then each block of its subgraph induced by the small vertices is a complete graph or an odd cycle.*

This result can be obtained directly from the proof of Erdős, Rubin and Taylor [62], too; however, the new proofs are much simpler. In fact, Gallai's original method [69] can also be applied. Moreover, for general graph properties $\mathcal{P}$, the variations [36, 146] of Brooks's and Gallai's theorems can be extended to list $\mathcal{P}$-colorings as well, see [35]. The corresponding result for hypergraphs appears in [132].

There are several results concerning 'critical amenable graphs,' too, where the lists are supposed to be nonidentical. See [39, 179] for further details and references.

Sparse graphs. Perhaps the most involved theorems concerning vertex choosability vs. vertex degrees are related to triangle-free graphs. The results summarized below are proved by a heavy use of probabilistic methods. The estimates nicely match with the general lower bounds on the independence number, in terms of order and maximum degree ([1, 166, 167, 168]).

Theorem 2.3. *Let G be any graph of maximum degree Δ.*

(i) If G has girth at least 5, then $\chi_\ell(G) \leq (1 + \epsilon_\Delta)\,\Delta/\log\Delta$, where $\epsilon_\Delta \to 0$ as $\Delta \to \infty$ (Kim [128]).

(ii) If G is triangle-free, then $\chi_\ell(G) \leq c\,\Delta/\log\Delta$ for some constant c independent of Δ (Johansson [113]).

(iii) For every $r \in \mathbb{N}$ there exists a constant c_r such that if G is K_r-free, then $\chi_\ell(G) \leq (c_r\,\Delta\log\log\Delta)/\log\Delta$ (Johansson [114]).

Apart from a multiplicative constant, the upper bounds in (i) and (ii) as functions of Δ are tight, since there exist graphs of arbitrarily large girth with maximum degree Δ and chromatic number $c\Delta/\log\Delta$ (see [24]). It remains an open problem to prove the asymptotic bound of (i) for the triangle-free case:

Conjecture 2.4. *(Kahn, Kim [128]) For triangle-free graphs G of maximum degree Δ,*

$$\chi_\ell(G) \leq (1 + o(1))\,\Delta/\log\Delta$$

as $\Delta \to \infty$.

For relatively small maximum degree $\Delta \geq 5$ and sufficiently large girth g with respect to Δ, the stronger explicit upper bound $\chi_\ell \leq \Delta/2+2$ was proved by Kostochka [129, Remark 6]. It follows, in particular, that every graph of maximum degree 5 and girth at least 35 is 4-choosable.

Almost disjoint lists. In the context of (p, q, r)-choosability, upper bounds in terms of vertex degrees have been derived by Kratochvíl, Tuza and Voigt [138]. For instance, it is shown by probabilistic methods that if the lists are almost disjoint (say, $r = p - c$) then lists of size $\sqrt{5.437\,c\,\Delta(G)}$ always admit a list coloring and this bound is best possible for all c, apart from a multiplicative constant.

2.2 Lower bounds on the choice number

The following result shows that χ_ℓ is closely related to the essentially *local* parameter of vertex degree. In this respect it essentially differs from the chromatic number which is a *global* graph invariant in nature.

Theorem 2.5. *(Alon [4]) Let $k \in \mathbb{N}$. If*

$$\overline{d} > 4\binom{k^4}{k}\log\left(2\binom{k^4}{k}\right)$$

holds for the average vertex degree $\overline{d} := \frac{1}{n}(d(v_1) + \ldots + d(v_n))$ of G, then $\chi_\ell(G) > k$.

The proof is probabilistic, performed in two main steps. Start with a bipartite subgraph $H \subseteq G$ of minimum degree at least $\overline{d}/4$, with vertex partition $A \cup B$, $|A| \geq |B|$. Simple calculation shows that selecting k-element lists for the vertices of B from a k^4-element color set $\mathbb{L}$ randomly and independently, each with probability $\binom{k^4}{k}^{-1}$, the probability that all k-subsets of $\mathbb{L}$ occur as lists in the neighborhood of a vertex v is at least $1/2$, for each $v \in A$. Call such a v good. We now fix a list assignment for B in which at least $|A|/2$ vertices $v \in A$ are good, and choose a k-subset of $\mathbb{L}$ for each good vertex of A, again randomly and independently. Since every coloring from the lists on B uses at least $k^4 - k + 1$ colors in the neighborhood of a good vertex, it is necessary for the colorability of H (and hence of G, too) that the list of each good $v \in A$ contains at least one of the remaining $k - 1$ colors. This can be seen to have probability less than k^{-2}, however, and thus by the independent random choice, any one of the $k^{|B|}$ possible colorings of B has an extension on A with probability *less than* $k^{-2|A|/2} \leq k^{-|B|}$. Consequently, some list assignment admits no coloring.

Approximability. One important consequence of Theorem 2.5 is that the choice number can be estimated within constant accuracy on every class of graphs where χ_ℓ is supposed to not exceed a fixed bound. This is obtained by observing the additional facts that the coloring number $\mathrm{col}(G)$ can be determined in linear time, and that every graph of average degree t contains a subgraph of minimum degree at least $t/2$.

Corollary 2.6. *There exists a function $h\colon \mathbb{N} \to \mathbb{N}$ and an algorithm $\mathcal{A}$ that finds, for every graph $G = (V, E)$, an $s \in \mathbb{N}$ in $O(|V| + |E|)$ steps such that $s \leq \chi_\ell(G) \leq h(s)$.*

Taking $h(s) = \mathrm{col}(G)$, the above method yields that $h^{-1}(x)$ can grow at least with the speed of $\log x / \log\log x$.

Unit distance graphs. Theorem 2.5 has several further interesting corollaries. One on graph unions will be discussed later, in Section 1.4. Here we mention a problem raised by Johnson [116] and solved by Jensen and Toft [112].

The *unit distance graph* in $\mathbb{R}^2$ has the points of the Euclidean plane as vertices, and two vertices are adjacent if and only if their distance in the plane equals 1. The chromatic number of this graph is known to be between 4 and 7. Jensen and Toft observe that the unit distance graph contains a d-regular bipartite subgraph for every $d \in \mathbb{N}$, namely the d-dimensional cube Q^d can be embedded into it (e.g., translating Q^i with a unit vector of general position, we obtain Q^{i+1}). Since $\chi_\ell(G_d)$ tends to infinity with d by Theorem 2.5, it follows that *the choice number of the planar unit distance graph is infinite.*

Making this assertion more precise, Schmerl [161] proved that the choice number for $\mathbb{R}^2$ and $\mathbb{R}^3$ is countable, as well as the 'rational distance graph' in $\mathbb{R}^2$; and that these bounds are not valid in higher dimension.

2.3 Graph polynomials

The *graph polynomial*, also called the *edge difference polynomial*, of a graph $G = (V, E)$ is defined as

$$P_G = P_G(x_1, \ldots, x_n) := \prod_{\substack{i<j \\ v_iv_j \in E}} (x_i - x_j)$$

for $E \neq \emptyset$. Assuming that the list assignment $\mathcal{L} = (L_1, \ldots, L_n)$ is given, the polynomials

$$Q_i = Q_i(x_i) := \prod_{q \in L_i} (x_i - q)$$

(for $i = 1, \ldots, n$) will also be of great importance.

The classical concept of graph polynomials was studied already in the 19th century, by Sylvester [173] and Petersen [152]. (For more recent references, see [7, 58].) In order to relate it to list colorings, Alon and Tarsi [7] first observe (by applying induction on n) that the following kind of 'Nullstellensatz' is valid.

Lemma 2.7. *Let $P(x_1, \ldots, x_n)$ be a polynomial of n variables over the ring $\mathbb{Z}$ of integers, and suppose that the degree of x_i in P is at most d_i. Let L_i be any subset of $\mathbb{Z}$ with cardinality $|L_i| = d_i + 1$, for $1 \leq i \leq n$. If $P(x_1, \ldots, x_n) = 0$ for all n-tuples $(x_1, \ldots, x_n) \in L_1 \times \cdots \times L_n$, then $P \equiv 0$.*

From this lemma, several useful results can be deduced for list colorings. The first one is an algebraic necessary and sufficient condition for list colorability.

Theorem 2.8. *(Alon, Tarsi [7]) A graph G with an n-tuple $\mathcal{L}$ of lists admits a list coloring if and only if the graph polynomial P_G does not belong to the ideal $\mathcal{I}(Q_1, \ldots, Q_n)$ generated by the polynomials Q_i.*

To prove this assertion (in either direction), one observes first that, substituting the color of v_i for x_i in P_G, the graph polynomial becomes zero if and only if the color assignment is not a proper coloring. On the other hand, choosing a color for each vertex from its list, makes the value of each Q_i equal to zero, therefore $x_i^{|L_i|}$ can be expressed in the form $\overline{Q}_i := \sum_{0 \leq j < |L_i|} c_{ij} x_i^j$ as long as we restrict ourselves to color assignments taken from the lists. Substituting $x_i^{|L_i|}$ by $\overline{Q}_i$ repeatedly in P_G, eventually we obtain a polynomial, say $\overline{P}_G$, in which each x_i has degree less than $|L_i|$, and furthermore $(P_G - \overline{P}_G) \in \mathcal{I}(Q_1, \ldots, Q_n)$. By the small degrees, $\overline{P}_G \in \mathcal{I}(Q_1, \ldots, Q_n)$ holds if and only if $\overline{P}_G \equiv 0$.

Now, if we assume that G admits no list coloring, then P_G as well as $\overline{P}_G$ is zero on the entire $L_1 \times \cdots \times L_n$, thus $\overline{P}_G \equiv 0$ and $P_G \in \mathcal{I}(Q_1,\dots,Q_n)$; and, conversely, assuming $P_G \in \mathcal{I}(Q_1,\dots,Q_n)$, we obtain $\overline{P}_G \equiv 0$, thus $P_G(x_1,\dots,x_n)=0$ for all $(x_1,\dots,x_n) \in L_1 \times \cdots \times L_n$, and therefore G admits no list coloring.

Uniquely list-colorable graphs. Dinitz and Martin [49] analyze irreducible factors of the remainder $\overline{P}_G$ of P_G modulo $\mathcal{I}(Q_1,\dots,Q_n)$, with emphasis on the case where G admits *precisely one* list coloring. For this purpose, it is convenient to view $\overline{P}_G$ as a homogeneous polynomial of degree $|E|$ over the set $\mathbb{L} \cup \{x_1,\dots,x_n\}$ of variables. (Note that the substituting operation with respect to the Q_i never destroys homogenity if also the colors are treated as variables.) It is proven in [49] that if $(G,\mathcal{L})$ admits precisely one coloring, say $(c_1,\dots,c_n)$, then $\overline{P}_G$ is the product of $|E|$ *linear factors*, where each x_i appears on the power $|L_i|-1$, and the other $|E|+|V|-\sum|L_i|$ factors are of the form $c_i - c_j$. What is more, E admits an orientation $\vec{E}$ for which a subset $\vec{E}' \subseteq \vec{E}$ can be chosen, $|\vec{E}'| = \sum_{i=1}^n (|L_i|-1)$, such that

$$\overline{P}_G(x_1,\dots,x_n) = \prod_{v_iv_j \in \vec{E}'} (x_i - c_j) \prod_{v_iv_j \in \vec{E}\setminus\vec{E}'} (c_i - c_j)\,.$$

Since $\overline{P}_G$ has degree less than $|L_i|$ in each variable x_i, and all but one colors are infeasible at v_i — setting P_G to zero for each $x_i = c$, $c \in L_i \setminus \{c_i\}$, the first product can be equivalently written as

$$\prod_{v_iv_j \in \vec{E}'} (x_i - c_j) = \prod_{i=1}^{n} \prod_{\substack{c\in L_i \\ c\neq c_i}} (x_i - c)\,.$$

In particular, the formula establishes a bijection between the edges of G and the irreducible factors of $\overline{P}_G$, for each uniquely colorable list assignment $\mathcal{L}$ on G.

2.4 Orientations and Eulerian subdigraphs

In general, it is not easy to check whether P_G can be expressed in terms of a combination of the Q_i with polynomial coefficients, therefore Theorem 2.8 is not a 'good characterization' in the algorithm-theoretic sense. One can deduce a more explicit sufficient condition for colorability from it, however, with the help of orientations. To formulate the result, call a digraph $\vec{G}$ *Eulerian* if the in-degree equals the out-degree for each of its vertices. (Hence, such a digraph is not required to be connected, and it is allowed to have an arbitrary number of isolated vertices, too.) We denote by $ee(\vec{G})$ the number of those *spanning* Eulerian subgraphs of $\vec{G}$ which have an even number of edges, and by $eo(\vec{G})$ the number of those with an odd number of edges.

Theorem 2.9. *([7]) Let a graph $G=(V,E)$ with a collection $\mathcal{L}$ of lists be given. If G has an orientation $\vec{G}$ such that the out-degree of each vertex v_i is at most $|L_i|-1$, and $ee(\vec{G}) \neq eo(\vec{G})$, then G is $\mathcal{L}$-list colorable.*

Call an orientation $\vec{G}$ of G *even* if it has an even number of edges v_iv_j with $i>j$, i.e., oriented from a vertex of larger subscript to a smaller one; and call $\vec{G}$ *odd* if the number of those backwards-oriented edges is odd. To prove Theorem 2.9, one observes first that, writing P_G as the sum of 2^m $(m := |E|)$ monomials, there is a bijection between those 2^m terms and the 2^m possible orientations of G. (In the factor $(x_i - x_j)$ of P_G, choose x_i if the edge v_iv_j is oriented from v_i to v_j, and choose $-x_j$ if it is oriented from v_j to v_i.) Hence, the monomials $\prod_{i=1}^m x_i^{d_i}$ are in one-to-one correspondence with those orientations in which the out-degree sequence is $(d_1,\dots,d_m)$. Thus, the coefficient of $\prod_{i=1}^m x_i^{d_i}$ in the standard representation of P_G equals the difference between the numbers of even and odd orientations having out-degree sequence $(d_1,\dots,d_m)$.

If two orientations $\vec{G}_1,\vec{G}_2$ have the same out-degree sequence, then the set $\vec{G}_1 \oplus \vec{G}_2$ of edges oriented differently in $\vec{G}_1$ and in $\vec{G}_2$ is an Eulerian subgraph, and the parity of the number of its edges is even if both $\vec{G}_1$ and $\vec{G}_2$ are even or both are odd, and the parity is odd if precisely one of $\vec{G}_1$ and $\vec{G}_2$ is even. Therefore, under the conditions of Theorem 2.9, the coefficient of $\prod_{i=1}^m x_i^{d_i}$ in P_G is nonzero (as the mapping $\vec{G}' \mapsto \vec{G} \oplus \vec{G}'$ is a parity-preserving bijection between orientations and Eulerian subgraphs if $\vec{G}$ is even, and parity-changing otherwise). Since all terms of P_G have degree m and every reduction step (by which $\overline{P}_G$ is derived from P_G) decreases the degree of the monomial to which it is applied, no new term $\prod_{i=1}^m x_i^{d_i}$ can occur during the reduction steps; and the original terms $\prod_{i=1}^m x_i^{d_i}$ in P_G are not modified because $d_i < |L_i|$ is assumed for all $1 \le i \le n$. Consequently, $\overline{P}_G \not\equiv 0$, and thus G admits a list coloring.

Orientations without odd circuits. An interesting case, worth mentioning separately, is where $eo(\vec{G}) = 0$, i.e., if no directed circuits of odd length occur in the orientation. Since $ee(\vec{G}) > 0$ (as the edgeless subgraph always is Eulerian), Theorem 2.9 implies that the maximum out-degree plus 1 is an upper bound on the choice number. For $eo(\vec{G}) = 0$, however, the algebraic machinery is not needed, as an elementary proof works by applying Richardson's theorem [153]. This result guarantees that, under the 'no odd circuits' assumption, G contains an independent set S such that from each $v \in V \setminus S$ there is at least one edge oriented to some vertex of S. In such orientations, the method of the proof described for Theorem 3.12 finds a list coloring whenever the out-degree of each vertex v_i is smaller than $|L_i|$.

In several situations, the following related observation turns out to be useful.

Lemma 2.10. *If $G=(V,E)$ and $d \in \mathbb{N}$ such that, for every $t \le |V|$, each induced subgraph on t vertices has at most dt edges, then G has an orientation of maximum out-degree at most d.*

This assertion seems to have been in the folklore at least from the second half of the 1980s; a proof can be found in [7]. By the observations above, if G is bipartite, then the lemma yields an orientation $\vec{G}$ with a guaranteed upper bound not only on the maximum out-degree, but also on the choice number.

Corollary 2.11. *Every 4-regular bipartite graph is 3-choosable. More generally, for all $k, m \in \mathbb{N}$, every $2k$-regular bipartite graph is $(km + m, m)$-choosable.*

Contrary to the algebraic proof of Theorem 2.9, these ideas can be turned to a polynomial algorithm that finds a list coloring when the relevant assumptions hold. On the other hand, as noted by Jensen and Toft [111], there seem to be no efficient algorithms known that find the smallest possible maximum out-degrees in orientations $\vec{G}$ with $ee(\vec{G}) \neq eo(\vec{G})$ or in those with no odd directed circuits.

4-regular Hamiltonian graphs. One of the successful applications concerns graphs with $3t$ vertices and $6t$ edges, whose edge set is the union of a Hamiltonian cycle and t vertex-disjoint triangles. For such graphs, Du and Hsu [52] conjectured that the independence number equals t, and Erdős raised the problem whether they always are 3-colorable. This has been answered in the following stronger form.

Theorem 2.12. *(Fleischner, Stiebitz [67]) If a directed graph $\vec{G}$ is the edge-disjoint union of a Hamiltonian circuit and some mutually vertex-disjoint, cyclically oriented triangles, then $ee(\vec{G}) - eo(\vec{G}) \equiv 2 \pmod 4$, and, consequently, the underlying graph of $\vec{G}$ is 3-choosable.*

Without applying the algebraic machinery of Theorem 2.9, Sachs [157] presents a purely combinatorial proof for the weaker assertion of 3-colorability.

List T-colorings. Recently, Alon and Zaks [9] generalized Theorem 2.9 for list-T-colorings. They consider multigraphs G^m where each edge of G is replaced by $2|T| - 1$ parallel edges if $0 \in T$, and by $2|T|$ parallel edges if $0 \notin T$. Then, if G^m admits an orientation $\vec{G}^m$ where $ee(\vec{G}^m) \neq eo(\vec{G}^m)$ and the out-degree of each vertex v_i is smaller than L_i, then G admits a list-T-coloring.

3 Comparisons of coloring parameters

In the bulk of this section, we investigate graph classes in which the choice number is not much larger than the chromatic number. Classical examples of this kind are the planar graphs, while a fundamental open problem is related to line graphs. At the end, we discuss the relationship between subset choosability and the fractional chromatic number.

3.1 Planar graphs

Planar graphs have always been special objects in the study of graph colorings. The paper by Erdős, Rubin and Taylor [62], too, contained several challenging questions about them. The answers (each found more than a decade later) and some further results are summarized next. One may note at the beginning that $\chi_\ell(G) \leq 6$ is easily seen, because every planar graph contains a vertex of degree at most 5, therefore $\mathrm{col}(G) \leq 6$ also holds.

Theorem 3.1.

(i) Every planar graph is 5-choosable (Thomassen [176]).

(ii) There exists a non-4-choosable planar graph (Voigt [191]).

(iii) Every planar graph is $(4, 1, 3)$-choosable (Kratochvíl, Tuza, Voigt [138]).

(iv) Every triangle-free planar graph is 4-choosable.

(v) There exists a non-3-choosable triangle-free planar graph (Voigt [193]).

(vi) Every triangle-free planar graph is $(3, 1, 2)$-choosable (Kratochvíl, Tuza, Voigt [138]).

(vii) Every planar graph of girth 5 is 3-choosable (Thomassen [177]).

(viii) Every bipartite planar graph is 3-choosable (Alon, Tarsi [7]).

Further constructions for parts (ii) and (v) were found by Gutner [82]. Moreover, as noted in [195], a construction of [82] (as well as one of [195]) is a non-4-choosable planar graph of chromatic number 3, having an uncolorable list assignment on as few as $|\mathbb{L}| = 5$ colors. The currently known smallest 3-colorable non-4-choosable planar graph, with 63 vertices, is presented by Mirzakhani [148] (also describing the interesting story of 'teamwork' how the record of 63 has been achieved). In her construction, too, an uncolorable list assignment with $|\mathbb{L}| = 5$ is given.

To (viii), one may note that $K_{2,4}$ is bipartite, planar, and not 2-choosable. Furthermore, the k-choosability results ($k = 3, 4, 5$) extend to (km, m)-choosability for all $m \in \mathbb{N}$. In connection with (iii), the following problem remains open.

Problem 3.2. *([138]) Is every planar graph $(4, 1, 2)$-choosable?*

Moreover, Škrekovski asks concerning (vi) whether there exist any planar, non-$(3, 1, 2)$-choosable graphs.

The proofs of the various upper bounds on the choice number in Theorem 3.1 use quite different techniques. Part (iv), that belongs to the folklore and seems to have been first mentioned explicitly in [136], is just a simple remark on applying Euler's formula; (vii) requires a lot of intermediate steps to verify; (iii), (vi), and (viii) are based on the fact that the graphs in question admit an orientation with maximum out-degree 3 and 2, respectively (cf. Lemma 2.10); and the proof of (i) is already a classic, that we present next.

The proof of 5-choosability. As the assertion is trivial for graphs of order at most 5, one can apply induction on n. We may assume that G is a 2-connected *near-triangulation*, i.e. all of its internal faces are triangles. Omitting colors from lists on the outer cycle C, the following induction hypothesis will be applied on the list assignments: Two consecutive vertices v_1, v_2 of C are colored (i.e., $|L_1| = |L_2| = 1$) with distinct colors, lists at the other vertices of C have size 3, and vertices not incident to C have lists of 5 colors each. If C is a triangle, one can immediately reduce G by omitting v_1, v_2, and their colors from the lists of their neighbors (with just a little more care if an internal vertex is adjacent to the entire C). Hence, we assume $|C| \geq 4$.

If C has a chord, say $v_i, v_j \in V(C)$ are adjacent but nonconsecutive on C, then $\{v_i, v_j\}$ splits G into two parts G_1, G_2, having the edge v_iv_j on their outer cycles, and one of them, say G_1, contains the two colored vertices of C. Finding a list coloring of G_1 by the induciton hypothesis, v_i and v_j get colored on the outer cycle of G_2, and then G_2 is also list colorable.

If C has no chord, consider the uncolored neighbor of v_2, say v_3, and reduce its list to a 2-element subset L'_3 not containing L_2. Since G is a near-triangulation and $|C| > 3$, the neighbors of v_3 induce a path P from v_2 to the uncolored neighbor v_4 of v_3 on C, and P is internally disjoint from C (as C has no chord); therefore, the lists of size 5 on P can be reduced to 3-element lists disjoint from L'_3. Finding a list coloring of $G - v_3$ by induction, v_4 is the unique vertex that can exclude one of the two colors from L'_3, therefore G, too, is list colorable.

Defective colorings. Cowen, Cowen and Woodall [45] consider vertex colorings φ which are not proper, but for a fixed $d \in \mathbb{N}$ every vertex v has at most d neighbors of color $\varphi(v)$. In the list coloring version, call a graph $(k,d)^*$*-choosable* if it admits such a coloring for every k-assignment $\mathcal{L}$. This concept was recently introduced independently and simultaneously by Škrekovski [170] and Eaton and Hull [55]. In the manuscript [170], the following collection of results is announced:

(i) Every planar graph is $(3,2)^*$-choosable.

(ii) Every triangle-free planar graph is $(3,1)^*$-choosable.

(iii) Every outerplanar graph is $(2,2)^*$-choosable.

(iv) Every triangle-free outerplanar graph is $(2,1)^*$-choosable.

The assertions (iii) and (iv) concerning outerplanar graphs have also been proved by Eaton and Hull. Both [170] and [55] ask whether every planar graph is $(4,1)^*$-choosable; if true, this would be an interesting generalization of a theorem of [45].

Answering a problem of [170] in the negative, Tuza and Voigt have constructed 3-colorable planar graphs which are not $(3,1)^*$-choosable. (One simple example is $27P_3 + 2K_1$.)

3.2 Graphs with equal chromatic and choice number

Beside the asymptotic results of Section 1.3, it would be of great interest to know which conditions ensure that the choice number equals the chromatic number. At the current state of the art, however, it seems hopeless to find a characterization theorem for graphs G satisfying $\chi_\ell(G) = \chi(G)$.

Graphs of small chromatic number. Already the case of 2-choosable bipartite graphs, settled by Rubin[1], is not at all trivial. To formulate the result, define the *core* of G the subgraph obtained by successively removing vertices of degree 1 as long as such a vertex is present in the current graph. Moreover, let us say for short that a graph is a *θ-graph* if it consists of two vertices of degree 3 joined by three paths of respective lengths $2, 2, 2m$ ($m \in \mathbb{N}$ arbitrary) all internal vertices of which have degree 2. (I.e., one edge of $K_{2,3}$ is subdivided into an odd path.)

Theorem 3.3. *(Rubin [62]) A connected graph is 2-choosable if and only if its core is either a single vertex or an even cycle or a θ-graph.*

The smallest uncolorable 2-assignments of a non-2-choosable graph require at most four colors in $\mathbb{L}$. Hoffman, Johnson and Wantland [99] observe that under the additional condition $|\mathbb{L}| \leq 3$, the graphs $K_{2,n-2}$ (and only those) become 2-choosable, for all n.

It is worth noting here that the T-choice version of list colorings seems to be much harder than the problem for $T = \{0\}$. Already for a subcase of $k = 2$, namely for cycles of even length, and for some rather restricted sets T, unexpected difficulties arise.

Conjecture 3.4. *(Alon, Zaks [9]) For every $n, r \in \mathbb{N}$, and for the set $T = T_r := \{0, 1, \ldots, r\}$,*

$$\chi_{\ell|T} C_{2n} = \left\lfloor \frac{4n-2}{4n-1} \cdot (2r+2) \right\rfloor + 1 .$$

In [9], the conjecture is proved for cycles of length four.

For a subclass of 3-colorable graphs, we mention the following result.

Theorem 3.5. *(Gravier, Maffray [75]) Suppose that $\omega(G) \leq 3$ in the graph $G = (V, E)$. If the edge set can be partitioned into two sets $E' \cup E'' = E$ in such a way that each induced P_3 of G has precisely one edge in each of E' and E'', then $\chi_\ell(G) = \chi(G) = \omega(G)$.*

Further problems. Graphs with larger chromatic number are considered in recent works by Gravier and Maffray. In [76] they investigate graphs in which there exists a k-coloring without color classes of more than 2 vertices. Corollaries are derived for *claw-free* graphs (i.e.,

[1] There are several important results in the paper [62] attributed by its authors to A. L. Rubin who was working on his Thesis at that time.

graphs containing no induced star of degree 3) of small order, from which it follows that *if G is the complement of a triangle-free graph, then $\chi(G) = \chi_\ell(G)$*. An interesting related problem is

Conjecture 3.6. *(Gravier, Maffray [75, 76]) If G is claw-free, then $\chi_\ell(G) = \chi(G)$.*

In some sense, this conjecture seems to be 'too strong,' and perhaps it would be worth making further efforts to find a counterexample. On the other hand, if it turns out to be true, then it implies the validity of the famous List Coloring Conjecture, too. (The latter will be discussed in the next subsection.)

Choice-perfect graphs. Motivated by the concept of perfect graphs, one can define various types of 'choice perfectness,' and raise the following problem.

Problem 3.7. *([181]) Characterize those graphs G in which $\chi_\ell(G') = f(G')$ holds for every induced subgraph G', where*

(i) $f(G') := \chi(G')$,

(ii) $f(G') := \omega(G')$.

The second property implies that G is perfect, but the first one doesn't; for instance, the odd cycles are 'perfect' in the sense of (i). Further choice-perfect classes will be mentioned in the next subsection.

Concerining the equality $\chi = \chi_\ell$, the following problem extends the famous Erdős–Lovász–Farber conjecture (see e.g. [60, p. 26]) for choosability.

Conjecture 3.8. *(Alon [5]) If G is the edge-disjoint union of n complete graphs of order n each, then $\chi_\ell(G) = n$.*

Kahn [124] has observed that a slight modification in the proof of the main result in [122] yields $\chi_\ell = n + o(n)$ for these graphs. From another point of view, motivated by Theorem 2.9, Alon and Seymour [5] have proved that such a graph always has an orientation with maximum out-degree at most $n - 1$.

A theorem on matroids. In the context of $\chi_\ell = \chi$, we mention the following result of Seymour [164] who derives it from the Matroid Union Theorem [57, 150]. Let M be a matroid whose set X of elements can be partitioned into k independent sets. If L_x is a set with $|L_x| \geq k$ for each $x \in X$, then there exists a partition of X into *independent sets* X_i, $i \in \bigcup_{x \in X} L_x$, such that $i \in L_x$ for all i and all $x \in X_i$.

It follows, in particular, that if the edge set of a graph can be decomposed into k forests, and each edge is assigned to a list of k colors, then a color can be chosen for each edge from its list so that no cycle is monochromatic. (Certainly, colorings obtained this way are usually not proper edge colorings.)

3.3 Edge and total colorings

There is a large number of results motivated by the List Coloring Conjecture (Conjecture 3.10 below) which states the equality $\chi = \chi_\ell$ for *line graphs*. In this subsection we survey the results related to this problem, but in the more convenient terminology of edge and total colorings, rather than coloring line graphs and total graphs.

We shall use the following notational conventions, analogously to vertex colorings.

Prime notation. The parameters corresponding to chromatic and choice numbers for *edge colorings* are denoted in the same way, except that we write χ' instead of χ, as follows:

$\chi'(G)$ = the *chromatic index* of G,

$\chi'_\ell(G)$ = the *edge choice number* or *list chromatic index* of G = the smallest k such that every k-assignment $\mathcal{L}$ on the edges of G admits a list coloring.

These parameters are just the corresponding values of $\chi(L(G))$ and $\chi_\ell(L(G))$ of the line graph $L(G)$ of G.

Double prime notation. The parameters for *total colorings* are denoted by χ'' with the analogous subscripts:

$\chi''(G)$ = the *total chromatic number* of G = the smallest number of colors in a proper coloring of $V \cup E$,

$\chi''_\ell(G)$ = the *total choice number* (or the total list chromatic number) of G = the smallest k such that every k-assignment $\mathcal{L}$ on $V \cup E$ admits a list coloring.

The following lemma, the variants of which have been observed by many authors, shows that total list colorings are closely related to the edge choice number.

Lemma 3.9. *For every graph G, $\chi''_\ell(G) \leq \chi'_\ell(G) + 2$.*

The key idea of the proof is to color the vertices first. This can be done, for any k-assignment with $k = \chi'_\ell(G) + 2$, by the inequalities $\chi(G) \leq \Delta(G) + 1 < \chi'_\ell(G) + 2$. Removing the vertex colors from the list of each edge, at least $\chi'_\ell(G)$ colors remain in each list, so that a total list coloring exists. In this way, every upper bound on the edge choice number yields one on the total choice number as well.

As accounted in [86], the following problem has been raised independently by several authors, including Vizing, Gupta, Albertson and Collins, and Bollobás and Harris.

Conjecture 3.10. ***(List Coloring Conjecture)*** *For every multigraph G, $\chi'_\ell(G) = \chi'(G)$.*

A challenging related recent problem has been raised by several authors independently (Borodin, Kostochka and Woodall [32]; Juvan, Mohar and Škrekovski [119]; Hilton and Johnson [95]). For general reference, we propose a name for it.

Conjecture 3.11. ***(Total Choice Conjecture)*** *For every multigraph G, $\chi''_\ell(G) = \chi''(G)$.*

Subdividing each edge of G into a path of lenght 2, we obtain a graph H whose square H^2 is isomorphic to the 'total graph' of G, so that $\chi(H^2) = \chi''(G)$ and $\chi_\ell(H^2) = \chi''(G)$. In this direction, Kostochka and Woodall [133] generalize the Total Choice Conjecture to the following one, that we may call the 'Square Choice Conjecture': *For every graph* G, $\chi_\ell(G^2) = \chi(G^2)$.

Though the List Coloring Conjecture is still open in general, considerable progress has been achieved. A trivial upper bound is $\chi'_\ell(G) \le \mathrm{col}(L(G)) < 2\Delta(G)$. After the subsequent improvements by Bollobás and Harris [26], Chetwynd and Häggkvist [41] (for triangle-free graphs), and Bollobás and Hind [27], Kahn [121, 122] proved the asymptotic result

$$\chi'_\ell(G) = \Delta + o(\Delta)$$

by the 'incremental random' method, not only for all graphs of maximum degree $\Delta(G) = \Delta$ as $\Delta \to \infty$, but also for families of *hypergraphs* of maximum degree Δ where each pair of vertices is contained in a sufficiently small number (i.e., $o(\Delta)$) of (hyper)edges with respect to Δ.

So far the estimate with best known error term for graphs seems to be

$$\chi'_\ell(G) = \Delta + O(\Delta^{2/3}\sqrt{\log \Delta}),$$

proved by Häggkvist and Janssen [87]. They also prove, by an involved application of Theorem 2.9, that

$$\chi'_\ell(K_n) \le n,$$

which is in fact best possible for n odd. A more restricted version of list total colorings of K_n, where the number of occurrences of the colors is also prescribed, is due to Sun [172] (proving a conjecture of [41]).

Line graphs of bipartite graphs. The following celebrated theorem settles Conjecture 3.10 for all cases where G is bipartite.

Theorem 3.12. *(Galvin [70]) If G is a bipartite multigraph, then $\chi'_\ell(G) = \chi'(G) = \Delta(G)$.*

If G has no multiple edges, then the surprisingly simple argument just combines the 'Stable Marriage Theorem' of Gale and Shapley [68] and a useful idea of Bondy, Boppana and Siegel [28], as follows. Start with a proper edge coloring $\varphi' : E \to \{1, \dots, \Delta(G)\}$. Denoting by X and Y the two vertex classes of G, for each incident edge pair e, e' with $\varphi'(e) > \varphi'(e')$, orient the edge $ee' \in E(L(G))$ from e to e' if $e \cap e' \in X$, and from e' to e if $e \cap e' \in Y$. In this orientation, the maximum out-degree is at most $\Delta(G) - 1$. Assuming that the out-degree of each e is smaller than the number of colors in the list of e (which is certainly the case at the beginning in any $\Delta(G)$-assignment), the following procedure successfully list-edge-colors G: Taking the colors $i \in \mathbb{L}$ one by one, consider the set $E_i \subseteq E$ of those uncolored edges whose lists contain i. By [68], E_i contains a matching M_i which is 'absorbant' in E_i, i.e., there exist edges oriented from each $e' \in E_i \setminus M_i$ to some $e \in M_i$. Assign color i to the members of M_i, remove i from the lists of $E_i \setminus M_i$, and delete all edges oriented from $E_i \setminus M_i$ to M_i. Since a list gets shortened only if the corresponding out-degree is decreased, all lists remain longer than the out-degrees, and eventually the entire G becomes edge-colored. For multigraphs, one needs an extension of the Stable Marriage Theorem, which follows immediately by a more general result of Maffray [143].

Earlier results and extensions. Galvin writes very modestly in his Introduction: "The proof is very simple and uses no new ideas." Nevertheless, his theorem settles the long-standing conjecture of Dinitz (raised in 1978, also cited in [62]) which is just the rather particular case $G = K_{n,n}$. Before Theorem 3.12, Janssen [110] solved the problem for all unbalanced complete bipartite graphs, proving $\chi'_\ell(K_{p,q}) = \max(p, q)$ for all $p \ne q$. (She proved that, with a suitably chosen out-degree sequence $\mathbf{d}$, $L(K_{p,q})$ admits just one orientation without cyclic triangles, while the even and odd orientations — cf. the first paragraph after Theorem 2.9 — containing at least one cyclic triangle can be matched with each other by a bijection. Consequently, the monomial corresponding to $\mathbf{d}$ in the standard representation of P_G has coefficient 1 or -1, implying list colorability. This idea was developed further in [87] for the proof of the upper bound $\chi'_\ell(K_n) \le n$ cited above, to match even and odd orientations which are not transitive on some clique in a fixed clique decomposition of a given graph.) Previous significant progress was achieved by Häggkvist [84], for the case $p \le 2q/7$. A self-contained presentation of the proof of Theorem 3.12 can be found in [171], and further sufficient conditions for list edge colorability (where the conditions on the edges are given by lists on the vertices, strongly motivated by problems on Latin squares) have been published by Häggkvist [85].

It is immediately seen that the $(m\,\Delta(G), m)$-choosability of the line graph of any bipartite multigraph G follows by the same argument for every $m \in \mathbb{N}$. Borodin, Kostochka and Woodall [32] extend this method to prove that if each edge $e = xy$ of G has a list of at least $\max(d(x), d(y))$ colors, then $L(G)$ admits a list coloring.

Note further that Galvin's theorem implies $\chi''_\ell(G) \le \Delta(G) + 2$ for every bipartite multigraph G. It is conjectured in [32] that a total list coloring exists already for edge-lists of length $\Delta(G) + 1$, provided that all vertices have lists of $\Delta(G) + 2$ colors. The converse (when only the vertex-lists are shortened to $\Delta(G) + 1$) always admits a list coloring, as shown above.

Nonbipartite multigraphs. Multiple edges seem to create lots of extra difficulties. Until quite recently, the only improvement on the trivial upper bound of 2Δ was Hind's unpublished inequality $\chi'_\ell \le 9\Delta/5$ in [97].

Theorem 3.13. *(Borodin, Kostochka, Woodall [32]) Let $G = (V, E)$ be a multigraph, and suppose that the list of each edge $e = xy \in E$ contains at least*

$$\max(d(x), d(y)) + \left\lfloor \tfrac{1}{2} \min(d(x), d(y)) \right\rfloor$$

colors. Then G admits a list coloring. In particular, $\chi'_\ell(G) \le \lfloor \frac{3}{2}\Delta(G) \rfloor$.

This result immediately implies Shannon's tight bound [165] on the chromatic index of multigraphs of given maximum degree. Moreover, $\chi''_\ell(G) \le \lfloor \frac{3}{2}\Delta(G) \rfloor + 2$ also follows. However, this bound may not be tight:

Conjecture 3.14. *([32]) If G is a multigraph of maximum degree $\Delta > 4$, then $\chi''_\ell(G) \le \lfloor \frac{3}{2}\Delta \rfloor$. Moreover, if G is connected, not complete and not an odd cycle, then every list assignment with edge-lists of size $\lfloor \frac{3}{2}\Delta \rfloor$ and vertex-lists of size Δ is colorable.*

List coloring analogues of several further questions can be raised, for instance whether $\chi'_\ell \le \Delta + \mu + 1$ where μ denotes the maximum edge multiplicity (conjectured in [119]), or whether $\chi'_\ell(G)$ does not exceed the largest of $\Delta(G)$ and

$$\Delta'(G) := \max\left\{ \tfrac{2|E(H)|}{|V(H)|-1} \;\middle|\; H \subseteq G,\ |V(H)| \text{ odd} \right\}$$

(cf. [122, p. 12]). Explanation for the latter formula is that for multigraphs G, the fractional chromatic index $\chi'^*_\ell(G)$ equals $\max\{\Delta(G), \Delta'(G)\}$, by the Matching Polytope Theorem of Edmonds [56] (cf. e.g. [162]). This bound is asymptotically valid:

Theorem 3.15. *(Kahn [123]) For the class of multigraphs G,*

$$\chi'_\ell(G) = (1 + o(1)) \max\{\Delta(G), \Delta'(G)\}$$

as $\Delta \to \infty$.

An attractive conjecture of Kahn [122, 123] states that the asymptotic equality of the edge choice number and the fractional chromatic index remains valid for r-uniform hypergraphs (or hypergraphs with maximum edge size r, possibly with multiple edges) as well, for every fixed $r \in \mathbb{N}$, as, say, χ'_ℓ gets large.

Some upper bounds on χ'_ℓ and χ''_ℓ in terms of Δ plus the *maximum local average degree* are presented in [32].

Regular graphs of class 1. Developing the algebraic method (cf. Sections 2.4 and 2.3) further, Ellingham and Goddyn [58] analyze the combinatorial meaning of the coefficients of the monomials in the expansion of the graph polynomial. Some of their results are summarized in the next theorem. In its second part, 'Kempe recoloring' means that in a proper edge coloring we interchange the two colors on a 2-colored cycle, and repeat this operation an arbitrary number of times.

Theorem 3.16. *(Ellingham, Goddyn [58]) Let G be a d-regular multigraph with $\chi'(G) = d$. If*

(i) G has an odd number of edge colorings with d colors, or

(ii) any two of its edge d-colorings have a Kempe recoloring to each other, or

(iii) G is planar,

then $\chi'_\ell(G) = d$.

The third part states that a d-regular planar multigraph has $\chi'_\ell(G) = d$ if and only if $\chi'(G) = d$. For the case of $d = 3$, this yields that the Four Color Theorem is equivalent also to the assertion that every planar 2-connected cubic graph is 3-edge-choosable. As noted in [58], this follows already from the results of Scheim [160], that can in turn be deduced by combining a theorem of Vigneron [188] (cf. also [107]) with some ideas of Alon and Tarsi [7].

Taking another view on graphs embedded in the plane, projective plane, torus, and the Klein bottle, Borodin, Kostochka and Woodall [32] provide sufficient conditions for the equalities $\chi'_\ell(G) = \Delta(G)$ and $\chi''_\ell(G) = \Delta(G) + 1$ in terms of combinations of girth and maximum degree, extending the earlier results and methods of Borodin [31]. The larger girth, the smaller vertex degree suffices. We recall here the case with unrestricted girth.

Theorem 3.17. *([32]) If a graph G of maximum degree $\Delta(G) \ge 12$ is embeddable in a surface of nonnegative characteristic, then $\chi'_\ell(G) = \Delta(G)$ and $\chi''_\ell(G) = \Delta(G) + 1$.*

The equalities $\chi'_\ell = \chi'$ and $\chi''_\ell = \chi''$ for *outerplanar* graphs have been proved by Juvan and Mohar [117].

The upper bound of $\Delta + 1$. Most of the results above verify the List Coloring Conjecture for some graphs with $\chi' = \Delta$. Concerning the other case, $\chi' = \Delta + 1$, Juvan, Mohar and Škrekovski study the problem for small maximum degree. They note that the upper bound $\chi'_\ell(G) \le 4$ for (simple) graphs with $\Delta(G) \le 3$ is implied by the choice version of the Brooks theorem (indeed, to create K_4 in a line graph would require a vertex of degree at least 4 or a triangle with a multiple edge), and prove in [118] the stronger assertion that if a subgraph $E' \subset E$ of maximum degree 2 has lists of size 3 and the edges of $E \setminus E'$ have lists of size 4, then G is list colorable. Subsequently, they prove in [118] that every graph of maximum degree 4 is 5-edge-choosable. Their method is strongly based on the treatment of so-called 'half-edges' (those incident to just one vertex), to which shorter lists are assigned, and so an inductive proof becomes possible by cutting off a suitably chosen small subgraph.

For unrestricted maximum degree, Kostochka [130] proved that if G contains no cycle shorter than $8\Delta(\log \Delta + 1.1)$, then $\chi'_\ell(G) \le \Delta + 1$.

3.4 Choice ratio and fractional chromatic number

Motivated by Problem 1.5, the study of the set

$$CH(G) := \left\{ \tfrac{k}{\ell} \;\middle|\; G \text{ is } (k,\ell)\text{-choosable} \right\}$$

leads to some interesting observations. It was first proved in Gutner's Thesis [81] (cf. also [4]) that the elements of $CH(G)$ can be arbitrarily close to $\chi(G)$.

The concept of fractional chromatic number admits a further strengthening in this assertion. Denote by $\mathcal{S}$ the collection of all independent sets in G, and consider

$$\chi^*(G) := \inf_{\varphi^*} \sum_{S \in \mathcal{S}} \varphi^*(S)\,,$$

where the infimum is taken over all functions

$$\varphi^* : \mathcal{S} \to \mathbb{R}^{\geq 0}$$

satisfying the condition

$$\sum_{\substack{S \in \mathcal{S} \\ v_i \in S}} \varphi^*(S) \geq 1$$

for every vertex $v_i \in V$. One can show that the infimum is in fact attained as minimum, and $\chi^*(G)$ — termed the *fractional chromatic number* of G — is a rational number.

Theorem 3.18. *(Alon, Tuza, Voigt [8]) For every graph G,*

$$\inf\{r \in CH(G)\} = \min\{r \in CH(G)\} = \chi^*(G)\,.$$

Choosing ℓ-element color sets $C_i \subseteq L_i$ from a k-assignment $\mathcal{L}$ of G, and defining $\varphi^*(S(j)) := 1/\ell$ for each $j \in \mathbb{L}$, where $S(j) := \{v_i \mid j \in C_i\}$, a fractional coloring of G with value k/ℓ is obtained, proving that $\chi^*(G)$ is a lower bound. The other direction for the infimum is not hard to prove by probabilistic methods ; and for the minimum it can be deduced from a theorem of Huckemann, Jurkat and Shapley (mentioned in [73] and proved also in [6]) by showing that for every fixed t and r, if the edge size of a uniform hypergraph with t edges is divisible by a suitably chosen integer, then the hypergraph admits a vertex partition of 'zero discrepancy' (i.e., equi-partitioning each edge) into r classes. This argument also yields that the minimum is attained for infinitely many pairs (k, ℓ). We note further that the result remains valid in a very general setting, for induced hereditary properties [147].

Theorem 3.18 yields that the implication given in Problem 1.5 is valid for infinitely many m, for every fixed pair (k, ℓ) with $k/\ell \in CH(G)$. Moreover, consequences for the 3-chromatic graph described in Conjecture 1.8 follow, too.

The sufficient value obtained from hypergraph theoretic methods for the smallest pair (k, ℓ) attaining $\chi^*(G)$ is rather large ; the next example shows that it can be the smallest one expected.

Example 3.19. *The equality $\chi^*(C_{2t+1}) = 2 + 1/t$ is easy to see. The following short argument shows that C_{2t+1} is $(2t+1, t)$-choosable for every $t \in \mathbb{N}$. Assuming that the vertices $v_1, \ldots, v_{2t+1}$ are labelled consecutively along the cycle, suppose that $\{1, 2, \ldots, 2t+1\} \subseteq \mathbb{L}$ in the $(2t+1)$-assignment $\mathcal{L}$, and that each color $j > 2t+1$ is missing from at least one list. Remove the color i from L_i if $i \in L_i$, and remove an arbitrary color otherwise. The consecutive occurrences of any one color induce subpaths P in the cycle. Select this color for the 1st, 3rd, 5th, ... vertices of P (proceeding clockwise), delete it from the 2nd, 4th, ... lists, and also delete the edges incident to a v_i when already t colors have been selected for v_i. Repeating this procedure for each color and each possible P sequentially, a subset of t colors will eventually be selected for every v_i because only those (at most) t colors get deleted from the shortened list of size $2t$ which have been selected for v_{i-1}.*

As regards bipartite graphs, Tuza and Voigt [185] showed that $K_{2,4}$ is $(2m, m)$-choosable if and only if m is even, and more generally they proved that the same property holds for every minimally non-2-choosable bipartite graph (unpublished, 1995).

3.5 The chromatic polynomial

Given a graph $G = (V, E)$ and a list assignment $\mathcal{L} = (L_1, \ldots, L_n)$, denote by $f(G, \mathcal{L})$ the number of $\mathcal{L}$-colorings $\varphi \colon V \to \mathbb{L}$. Kostochka and Sidorenko [131] proposed the problem of studying the function

$$F(G, k) := \min_{|L_1| = \ldots = |L_n| = k} f(G, \mathcal{L})\,,$$

i.e., the minimum number of $\mathcal{L}$-colorings taken over all k-assignments $\mathcal{L}$. (The maximum would obviously be k^n, for all $k \in \mathbb{N}$.)

Denoting by $P(G, k)$ the *chromatic polynomial* of G, it is clear by definition that $F(G, k) \leq P(G, k)$ holds for every G and every k, and the non-k-choosable k-chromatic graphs show that in some cases this inequality is strict.

Theorem 3.20. *(Donner [50]) For every graph G there exists an integer $k_0 = k_0(G)$ such that*

$$F(G, k) = P(G, k)$$

holds for all integers $k \geq k_0$.

The starting point of the proof is an observation that allows us to compute $f(G, \mathcal{L})$ recursively for every $\mathcal{L}$ (not only for k-assignments). For any $e = v_i v_j \in E$, denote by G/e the graph obtained by contracting e (i.e., replacing v_i and v_j by a new vertex v' and joining v' to each vertex adjacent to at least one of v_i and v_j) and $G - e := (V, E \backslash \{e\})$. For G/e, define the list assignment $\mathcal{L}/e$ to be identical to $\mathcal{L}$ on $V \setminus \{v_i, v_j\}$, and $L_{v'} := L_i \cap L_j$ for the contracted vertex. One can see that that

$$f(G, \mathcal{L}) = f(G - e, \mathcal{L}) - f(G/e, \mathcal{L}/e)$$

holds for all G, $\mathcal{L}$, and $e \in E$. To prove $f(G, \mathcal{L}) \geq P(G, k)$ for every k-assignment $\mathcal{L}$, Donner considers a computation tree based on the above recursion, and makes estimates on the values at its leaves (each leaf is an edgeless graph). The partial sums of those values are analyzed by distinguishing between the leaves according to the number of contractions on the computation tree from the root to the leaf in question.

Problem 3.21. *For which graphs G is the function $F(G, k)$ identical to the chromatic polynomial $P(G, k)$?*

Kostochka and Sidorenko [131] have observed that this equality holds for all *chordal* graphs; on the other hand, it obviously does not hold for any G with $\chi_\ell(G) > \chi(G)$. In the latter case, it follows by Donner's theorem that $F(G,k)$ *is not a polynomial*. (Since $F(G,k)$ and $P(G,k)$ coincide on all sufficiently large values of k, the former is a polynomial if and only if it is identical to the latter.)

4 Algorithmic complexity

In this section we discuss some algorithmic results. For terminology not introduced here concerning algorithmic complexity, we refer to [71] or the more recent book [37].

Note first that, since CHROMATIC NUMBER is a particular case of LIST COLORING (as well as of PRECOLORING EXTENSION), in general the NP-completeness of the latter follows from that of χ immediately. On the other hand, though the reductions presented at the beginning of Section 1 imply that these problems are equally hard as long as the class of all graphs is considered, this is not necessarily the case anymore for many nicely structured subclasses.

For convenience, let us formulate the algorithmic questions as decision problems. Keeping previous notation, the vertex set will be assumed to be $V = \{v_1, \ldots, v_n\}$ throughout. We shall first consider

PRECOLORING EXTENSION (PREXT) :

Instance: Graph $G = (V, E)$, subset $W \subseteq V$ of *precolored* vertices, *precoloring* $\varphi_W : W \to \mathbb{N}$, *color bound* k.

Question: Does there exist a proper coloring φ with at most k colors such that $\varphi(v) = \varphi_W(v)$ for all $v \in W$?

For lists of equal size, the problem is

k-LIST COLORING (k-LC) :

Instance: Graph $G = (V, E)$, *list assignment* $\mathcal{L} = (L_1, \ldots, L_n)$, with $|L_i| = k$ for all $1 \le i \le n$.

Question: Does $\mathcal{L}$ admit a list coloring on G?

The general case, where no restriction is put on the lengths of the lists, will be called LIST COLORING, abbreviated LC.

k-CHOOSABILITY (k-CH) :

Instance: Graph $G = (V, E)$.

Question: Does G have a list coloring for *every* k-assignment $\mathcal{L}$?

Obviously, the first two problems belong to the class NP. On the other hand, it will turn out that k-CH is located higher in the hierarchy of complexity classes. (A well known fundamental open problem is whether or not those types of complexity are indeed distinct.)

We shall proceed in the order of increasing difficulty, considering PREXT first, also presenting the known transparent necessary and sufficient conditions; the complexity of k-LC and the results related to k-CH will be discussed in the third and fourth subsections. Finally, we shall discuss results on graph coloring games.

Before the results on restricted graph classes, we quote a theorem on the running time of general list coloring algorithms.

General upper bounds. The chromatic number of a graph is a hard-to-estimate parameter, and all known algorithms determining it exactly run in exponential time with respect to the number n of vertices (even when the graph in question is supposed to be 3-colorable). In particular, Lawler [140] proposes an inductive algorithm that computes the chromatic number of G and of all its induced subgraphs, where the total number of steps is bounded above by $(\sqrt[3]{3}+1)^n$ times a polynomial of n. The method is based on the theorem of Moon and Moser [149] who proved that no graph of order n can have more than $3^{n/3}$ independent sets maximal under inclusion. (One also needs the fact that the maximal independent sets can be listed efficiently, see [180, 115].) Variants of this result, e.g. those in [104] and [65], enable us to improve on the guaranteed running time of coloring algorithms when restricted classes of graphs are considered. What is more, Lawler's method can be extended for list colorings as well, and the following result is valid.

Theorem 4.1. *(Hujter, Tuza [106]) There exists a polynomial $p(x)$ and an algorithm $\mathcal{A}$ such that, for every graph $G = (V, E)$ and every list assignment $\mathcal{L}$,*

(i) the algorithm $\mathcal{A}$ decides in at most $p(|V|)\cdot|\mathbb{L}|\cdot(\sqrt[3]{3}+1)^{|V|}$ steps whether or not G is list colorable;

(ii) if G is triangle-free, then $(\sqrt[3]{3}+1)^{|V|}$ can be replaced by $(\sqrt{2}+1)^{|V|}$ in the upper bound;

(iii) and, for every fixed $t \in \mathbb{N}$ and $\varepsilon > 0$, there is an $n_0 = n_0(t, \varepsilon)$ such that $(\sqrt[3]{3}+1)^{|V|}$ can be replaced by $(1+\varepsilon)^{|V|}$ for every graph of order $|V| \ge n_0$ that contains no induced matching of t edges.

The above bounds are similar to those for the chromatic number, the only difference is the (necessary) presence of the factor $|\mathbb{L}|$.

4.1 Precoloring extension

Below we summarize the known results, grouped according to graph classes. To make more sensitive distinctions, in some cases we shall impose restrictions on the *precolored* set W, too. For convenience, we shall assume that the monochromatic subsets of W are $W_1, W_2, \ldots, W_k$ (some of them may be empty), and that they are labelled in a decreasing order of cardinality, $|W_1| \ge \ldots \ge |W_k|$. The case of $W_1 = \emptyset$ leads to the complexity of CHROMATIC NUMBER, the literature of which will not be surveyed here; i.e., we assume $|W_1| \ge 1$ throughout. Unless otherwise stated, the given time complexity refers to the original PrExt problem; 'linear' means $O(|V|+|E|)$. The graph is said to be *F-free* if it contains no *induced* subgraph isomorphic to F.

Bipartite graphs: NP-complete in general [103], also for $|W|=3$ [22], on *planar* bipartite graphs with $k=3$ and on P_{14}-free bipartite graphs with $k=5$ [134], P_6-free bipartite graphs with unbounded k [105]; linear if $k=2$ (trivial), on P_5-free bipartite graphs for any k [105], and on trees and forests [105, 109].

Line graphs: NP-complete on line graphs of complete bipartite graphs [42]; polynomial on line graphs of multiforests [145].

Split graphs and complements of bipartite graphs: polynomial, of the same complexity as Bipartite Matching, apart from a multiplicative constant [105] (fastest known algorithms of $O(n^{2.5})$, see e.g. [101]).

Interval graphs: $O(n^3)$ if $|W_1|=1$, and NP-complete if just $|W_1|=2$ is assumed [19].

P_4-free graphs (cographs): linear [17, 105, 109].

Permutation graphs: NP-complete, already for $|W_1|=1$ [108].

Complements of Meyniel graphs: polynomial if $|W_1|=1$ [105], by the results of Hertz [93], applying the algorithms of Grötschel, Lovász and Schrijver [78, 79]. (A graph is said to be a *Meyniel graph* if each of its odd cycles of length ≥ 5 contains at least 2 chords.)

Perfect graphs: polynomial if $W_3=\emptyset$ and $|W_2|\leq 1$, and NP-complete otherwise [135].

The NP-completeness for $|W_3|\geq 1$ or $|W_2|\geq 2$ on perfect (more explicitly, on bipartite) graphs follows immediately from the results of [22, 134] for $k=3$. On the other hand, as mentioned in [105], the complexity of PrExt is not known for several graph classes whose structure is well understood, e.g. for *unit interval graphs*; neither PrExt with the additional condition $|W_1|=1$ for *chordal* (and, in particular, *strongly chordal*) graphs. Here is another innocent-looking related problem:

Conjecture 4.2. (Woeginger [197]) *On planar bipartite graphs,* PrExt *with $k=3$ and $|W_1|=|W_2|=|W_3|=1$ is solvable in polynomial time.*

Woeginger notes that the condition $|W_1|=1$ makes the problem straightforward to solve on this restricted class for any other color bound. The polynomial instances will be discussed further in the next subsection, where structural characterizations will be given for the extendability of precolorings.

Distance constraints on W. Thomassen [178] proved for *planar* graphs G that if $k\geq 5$ and the vertices of W are sufficiently far apart (with respect to $|W|$), then every k-coloring of W can be extended to that of the entire G. This result has recently been strengthened considerably by Albertson [2], proving that a percoloring is extendable in either of the following cases:

(i) $k>\chi(G)$ and the distance between any two precolored vertices is at least 4;

(ii) $k>\chi_\ell(G)$ and the distance between any two precolored vertices is at least 3.

In particular, in a planar graph, distance 4 and 3 suffices for the extendability of a partial 5-coloring and 6-coloring, respectively. Albertson proves analogous results for the more general case, too, where W induces the union of vertex-disjoint cliques of sufficiently large mutual distances.

One of the interesting questions raised in [2] is whether or not distance constraints have similar consequences for list colorings. That is, if W is precolored, lists of given length $k>\chi_\ell$ are associated to the precolorless vertices, and we wish to extend the precoloring of W to a coloring of the entire graph by choosing a color from each list, how large should then be the distances between the vertices of W? In particular, what is the smallest distance (if any) that suffices for planar graphs and lists of length 5?

Undecidable problems. Here we mention some results on *infinite* graphs. Similarly to the finite case, one can ask whether a given precoloring on a finite subgraph is extendable to a proper k-coloring of the entire graph, with fixed color bound k.

Burr [40] investigates this problem for a class of graphs of fairly transparent structure, called *doubly-periodic* graphs. The vertices of such a graph G are labelled $v_{ij\ell}$ ($i,j\in\mathbb{Z}$, $\ell\in\{1,\ldots,n\}$), the subgraphs induced by $\{v_{ij1},v_{ij2},\ldots,v_{ijn}\}$ — called cells — are isomorphic for all pairs i,j, any other edge joins neighboring cells (i.e., cells whose i and j differ by at most one), and both mappings $i\mapsto i+1$ and $j\mapsto j+1$ are automorphisms of G.

It is proved in [40] that, for every color bound $k\geq 3$, there exists a doubly-periodic planar graph G of maximum degree 4 and a finite precolored set such that it is undecidable whether the precoloring can be extended to a k-coloring of G. Dukes, Emerson and MacGillivray [53] generalize this result to homomorphisms $G\to H$ (Burr's theorem deals with $H=K_k$). They prove undecidability for every finite, non-bipartite H, and for several finite bipartite graphs H, too; e.g., for H containing a cycle C of length at least 6, such that there is a homomorphism $h\colon H\to C$ with $h(v)=v$ for all vertices v of C. It remains open, however, to characterize which H make the problem undecidable (and, in particular, to prove or disprove decidability if H is a tree).

4.2 Good characterizations

There are some transparent conditions that can be checked efficiently on fairly large graph classes and provide good characterizations for the polynomial instances listed above. Most of them are collected in the paper by Hujter and Tuza [105]; and an efficient general method for perfect graphs with restricted precolorings has been developed by Kratochvíl and Sebő [135].

Core Condition. A nonempty set U of pairwise adjacent precolorless vertices is called a *q-core* if there are at least $q - |U|$ distinct monochromatic classes $W_i \subseteq W$ such that each vertex $u \in U$ has at least one neighbor in each of those W_i. If $|U| = 1$, then U is also called an *elementary* q-core. The *Core Condition* requires that the precoloring of G contains no $(k+1)$-core.

Sequence Condition. Starting with a partial k-coloring of G, repeat the following procedure until it terminates. If there is a $(k+1)$-core or there exists no elementary k-core, then stop; otherwise choose an elementary k-core $\{u\}$, and assign to it the unique color not appearing in its neighborhood. The *Sequence Condition* requires that such a procedure must not result in a $(k+1)$-core.

Independence Condition. For each precolored class W_i and each precolorless vertex set U, denote by $\alpha(U, i)$ the largest number of those mutually nonadjacent vertices in U which have no neighbor in W_i. The *Independence Condition* requires $|U| \leq \sum_{i=1}^{k} \alpha(U, i)$ for all $U \subseteq V \setminus W$.

It is easily seen that each of the above conditions is necessary for the extendability of a precoloring if the color bound is k. The next statement summarizes the known cases where they are sufficient as well.

Theorem 4.3. *For the extendability of any partial coloring with color bound k in an instance of* PrExt,

(i) The Core Condition is necessary and sufficient for split graphs, complements of bipartite graphs, P_4-free graphs, and, if no color is repeated in W, then also for complements of Meyniel graphs.

(ii) The Sequence Condition is necessary and sufficient for forests, and, if $k = 2$, then also for bipartite graphs.

(iii) The Independence Condition is necessary and sufficient for line graphs of multiforests.

Part (iii) has been re-stated from the paper by Marcotte and Seymour [145], the other results appeared in [105]. The case of interval graphs, with the assumption that no color is repeated in the precoloring, admits a characterization in terms of a Menger-type condition on directed graphs (constructed from the corresponding instance of PrExt); see [17, 105] for details.

PrExt-perfect graphs. Motivated by the Core Condition, the following graph operation can be introduced. Let $\mathcal{G}$ be a graph class closed under induced subgraphs. For each $G \in \mathcal{G}$ and for each (proper) partial k-coloring of G, contract each precolored color class to one new vertex, and make those new vertices mutually adjacent. The class of graphs obtained in this way from $\mathcal{G}$ will be denoted by $\mathcal{G}^*$. It has been observed in [105] that if every $G \in \mathcal{G}$ is perfect, and for every precoloring of every $G \in \mathcal{G}$ the core condition is sufficient for precoloring extendability, then every $G^* \in \mathcal{G}^*$ is perfect, too. Perfect graphs satisfying this requirement are called *PrExt-perfect* in [105]. Their characterization — as well as that of the corresponding class obtained by contraction — remains an open problem.

One of the interesting cases is the class $\mathcal{G}$ of P_4-free graphs (cographs). Recently, Van Bang Le [141] described $\mathcal{G}^*$ for them. It follows, in particular, that the membership in this class can be decided in polynomial time. (The cographs themselves can be recognized in linear time, see [43].) A characterization in terms of forbidden subgraphs, however, is not known so far.

Good characterization for PrExt on perfect graphs. We close this subsection with the strongest known related result on the general class of perfect graphs. For a vertex $v \in V$ and a collection $\mathcal{H}$ of not necessarily distinct subsets of V, $d_{\mathcal{H}}(v)$ denotes the number of those sets in $\mathcal{H}$ which contain v. The term 'ω-clique' means 'complete subgraph on $\omega(G)$ vertices.'

Theorem 4.4. *(Kratochvíl, Sebő [135]) Let $G = (V, E)$ be a perfect graph and $X, Y \subseteq V$ two disjoint independent sets. Then G has a proper coloring $\varphi\colon V \to \{1, \dots, \omega(G)\}$ with the properties that X is monochromatic and $\varphi(y) \neq \varphi(X)$ for all $y \in Y$, if and only if*

$$|\mathcal{Q}| \geq |\mathcal{K}| + |X|$$

holds for every multi-family $\mathcal{Q}$ of cliques and every family $\mathcal{K}$ of at most $|V|$ distinct ω-cliques satisfying

$$d_{\mathcal{Q}}(v) = d_{\mathcal{K}}(v) \qquad \forall\, v \in V \setminus (X \cup Y)$$

and

$$d_{\mathcal{Q}}(v) = d_{\mathcal{K}}(v) + 1 \qquad \forall\, v \in X\,.$$

The polynomial-time algorithm finding a required coloring when it exists is combinatorial, except for the only part that it calls for a maximum clique (for which no combinatorial algorithm of polynomial running time is known so far on perfect graphs). For the particular case of $Y = \emptyset$, this result answers a problem of Seymour who proved that it is NP-complete to decide whether two independent sets X, Y of unrestricted cardinalities in a perfect graph admit a proper coloring with $\omega(G)$ colors such that X and Y are contained in distinct color classes [163].

4.3 List colorings

On dense graphs, even with a very transparent structure, the List Coloring problem is quite hard. In fact, as Jansen and Scheffler [109] prove, it is NP-complete already on *complete bipartite graphs*, despite it is solvable in linear time on every graph without induced subgraphs P_4 if the total number $|\mathbb{L}|$ of colors is bounded. Also, Kubale [139] observes that the NP-completeness of LC on line graphs of complete graphs follows from that of the Chromatic Index problem [100]. (In [139], LC is shown to be NP-complete for bipartite graphs, too, under the further restriction that $|\mathbb{L}| = 5$ holds.) Recently, Jansen [108] proved NP-completeness for the union of two complete graphs. It is a natural related question to investigate which are the *sparsest* hard instances for LC.

Polynomially solvable cases. In both early papers [190, 62] it is observed that 2-LC is easy to solve. Indeed, one can obtain a linear-time algorithm by simply guessing the color $\varphi(v)$ of a vertex v and check what sort of implications this color would have for the other vertices. If $\varphi(v)$ occurs in the list of some neighbor u of v, then u gets forced to be asssigned to the other color of its list; and this forcing step may be repeated for the neighbors of u, etc. If this procedure stops when a subgraph G' is properly colored while all uncolored vertices still have two colors in their lists, then G is list colorable if and only if so is $G - G'$. On the other hand, if $\varphi(v)$ leads to a contradiction (excluding both colors from the list of some vertex), then in any list coloring of G (if it exists), the only choice for v can be the other color, which then either leads to a final contradiction or reduces the problem to a smaller subgraph in linear time.

Further easy instances include the graphs of maximum degree 2, as well as those list assignments (with arbitrarily long lists) where each color occurs in at most two lists.

Sparse hard instances. The above examples show that the following result is tight in several ways.

Theorem 4.5. *(Kratochvíl, Tuza [136]) The* List Coloring *problem is NP-complete when restricted to the instances where each list contains at most* 3 *colors, each color occurs in at most* 3 *lists, and G is a planar bipartite graph of maximum degree* 3.

This result is proved by applying one of the several connections between LC and the Satisfiability problem. Given a Boolean formula Φ in conjunctive normal form, with a set C of clauses over the set X of variables, one can define a graph G_Φ with vertex set $V = X \cup C$ and edge set

$$E := \{xc \,|\, x \in c \in C \text{ or } \neg x \in c \in C\}\,.$$

The symbols x and $\overline{x}$ $(x \in X)$ will be taken for the colors, and the lists $L(x)$ and $L(c)$ for the variable vertices x and clause vertices c will be defined as

$$L(x) := \{x, \overline{x}\} \qquad \forall\, x \in X$$

and

$$L(c) := \{\overline{x} \,|\, x \in c\} \cup \{x \,|\, \neg x \in c\} \qquad \forall\, x \in X\,.$$

It can be seen that there is a one-to-one correspondence between the satisfying truth assignments of Φ and those color assignments of X which can be extended to a list coloring of G_Φ. (Choose color $\overline{x}$ for a variable vertex x if and only if the variable x is false in the truth assignment; and, conversely, let x be false in Φ if and only if the color $\overline{x}$ has been chosen for x in a list coloring of G_Φ.) Hence, each list coloring uniquely determines a truth assignment, but not vice versa, because in some truth assignments some clauses are satisfied by more than one variable, each allowing a distinct color choice.

By this construction, the various theorems on Satisfiability (e.g., on 3-SAT) yield NP-completeness results on list colorings restricted to the corresponding graph classes. Note further that edges may be added to G_Φ in an arbitrary way as long as it remains bipartite, and still the two-way mapping between colorings and truth assignments is preserved. It follows, for instance, that LC is NP-complete on 3-regular bipartite graphs.

Note that also the degree condition in Theorem 4.5 is quite strong when compared to the chromatic number problem. In fact, by applying the theorem of Brooks, we obtain that $\chi(G) = 3$ can be decided in linear time for graphs of maximum degree 3, since $\chi(G) \le 3$ holds if and only if G contains no connected component isomorphic to K_4.

Colors in a bounded number of lists. For longer lists, the easy and hard instances can be separated in terms of bounds on the number of how many times a color may appear in the lists. Define the following problem class for $k, d \in \mathbb{N}$:

(k, d)-LC:

Instance: Graph $G = (V, E)$, list assignment $\mathcal{L} = (L_1, \ldots, L_n)$, $|L_i| = k$ for all $1 \le i \le n$, each color appearing in at most d lists.

Question: Does $\mathcal{L}$ admit a list coloring on G?

Theorem 4.6. *([136]) Let $k \ge 3$, d arbitrary.*

(i) If $d \le k$, then every instance of (k, d)-LC admits a list coloring, and a feasible coloring can be found in $O(n^{2.5})$ steps.

(ii) If $d > k$, then (k, d)-LC is NP-complete.

The first part of this result means that the colorability does not depend on the actual structure of the graph in question; i.e., one may assume $G = K_n$. In this case, a list coloring exists if and only if the lists admit distinct representatives, and therefore the problem is equally hard as Bipartite Matching (or, more explicitly, as finding a matching that covers the smaller vertex class of a bipartite graph).

Hall Condition. The following concept may be viewed as the LC-analogue of the Independence Condition given in Section 4.2. For graph $G = (V, E)$, list assignment $\mathcal{L}$, subset $U \subseteq V$, and color $i \in \mathbb{L}$, denote by $\alpha(U, i)$ the largest size of an independent set in the subgraph induced by those vertices of U whose lists contain color i. It is obvious that the condition

$$|U| \le \sum_{i \in \mathbb{L}} \alpha(U,i) \qquad \forall\, U \subseteq V$$

is necessary for the existence of a list coloring. Hilton and Johnson [94] and Gröflin [77] prove that this coindition is *sufficient for all $\mathcal{L}$ on G if and only if each 2-connected component of G is a complete subgraph.* In the particular case of *line graphs* $G = L(H)$, the necessary and sufficient condition is that H should be a forest (de Werra [48]). If multiple edges are also allowed, the Hall Condition becomes sufficient on multiforests if we require that any two parallel edges have the same list (Marcotte and Seymour [145]).

It may be noted that if all blocks are cliques, a polynomial list coloring algorithm can be designed even without the above characterization at hand. For this, one can take an endblock K sitting on a cut vertex v_j, and check for each color $i \in L_j$ one by one whether $\varphi(v_j) = i$ can be extended to a list coloring on the entire K. (This amounts just to testing whether the $L_{j'} \setminus \{i\}$ in $K \setminus \{v_j\}$ have distinct representatives.) Restricting L_j to those colors which do, the problem gets reduced to the subgraph induced by $V \setminus (V(K) \setminus \{v_j\})$ which is list colorable with the modified L_j if and only if so is the entire G with the original lists.

Hall number. An interesting related graph invariant, introduced and studied recently by Hilton, Johnson and Wantland [96, 95], is the *Hall number*, defined as the smallest natural number k such that the Hall Condition ensures colorability for every list assignment $\mathcal{L}$ with $|L_i| \ge k$ for all vertices of the graph in question. Obviously, the Hall number cannot be larger than the choice number. In the forthcoming papers [96, 95] the Hall number is compared to some other important parameters, too, such as the chromatic number and the independence ratio. Its irregular behavior is analyzed as well, by showing that the removal of a vertex or an edge may cause a rather large decrease or increase, respectively. The best possible results concerning these 'jumps' with respect to vertex degrees have been obtained by Tuza [181], showing that the Hall number of $K_n - e$ is equal to $n-2$ (while it is 1 for both K_n and K_{n-1}).

Subset choosability. Concerning the more general problem of choosing subsets that are disjoint if the corresponding vertices are adjacent, the concept of (p,q,r)-LC is defined in the natural way, with instance $G = (V,E)$ together with lists L_i of cardinality p each and $|L_i \cup L_j| \ge p + r$ if $v_i v_j \in E$, and the quesiton is whether q-element subsets can be chosen that are disjoint if the corresponding vertices are adjacent. The complexity of this problem is completely characterized:

Theorem 4.7. *(Kratochvíl, Tuza, Voigt [137]) The (p,q,r)-LC problem is NP-complete for*

$$p \ge \max\{q+2, r+1\}$$

and solvable in linear time for $p = r \ge q$ and for $q \le p \le q+1$.

Graphs of bounded treewidth. One of the equivalent definitions of *treewidth* is introduced in terms of chordal graphs:

$$tw(G) := \min\{\omega(H) - 1\},$$

where the minimum is taken over all chordal graphs H containing G as a subgraph. Representing such an H as the intersection graph of subtrees $T_1, \ldots, T_n$ of a tree T, the sets $X_z := \{v_i \in V \mid y \in V(T_i)\}$ $(z \in V(T))$ together with T form a so-called *tree decomposition* of G, a very convenient structure for algorithmic purposes. On this basis, for many NP-complete problems there exist polynomial (and often even linear) algorithms when restricted to graphs of treewidth less than t (often called *partial t-trees*), $t \in \mathbb{N}$ fixed; see e.g. [12, 44]. The methods of dynamical programming can be applied successfully for list colorings as well:

Theorem 4.8. *(Jansen, Scheffler [109]) Let $t \in \mathbb{N}$ be fixed. Then, on the class $\mathcal{G}_t$ of graphs of treewidth less than t, the* List Coloring *problem is solvable in $O(n^{t+2})$ time. Moreover, for every fixed $k \in \mathbb{N}$, if a tree decomposition of width $< t$ is given for any $G \in \mathcal{G}_t$, then* LC *is solvable in $O(n)$ time.*

A stronger time bound can be proved for trees. Since both the choice and coloring number of every tree with at least one edge equals 2, one can solve LC (similarly to PrExt) on trees in linear time without assuming any bound on the total number of colors, firstly coloring the vertices whose list consists of just one color, then deleting those colors from the lists of the neighbors and continuing this procedure as long as 1-element lists occur. If no list becomes empty at the end of this phase, then each uncolored component is an instance of 2-LC (we may delete colors from the lists longer than 2), and can be list-colored in linear time, by choosing an arbitrary root with any color from its list and then proceeding from the root towards the leaves e.g. by breadth-first search. In this way, not only the decision problem but also the search version is solvable in linear time. Jansen and Scheffler also note that the *number* of admissible list colorings can be determined in $O(kn)$ time, where k denotes the total number of colors.

Cardinality-constrained color classes. Let $G = (V,E)$ with the list assignment $\mathcal{L}$ be given, and suppose that for each $i \in \mathbb{L}$ an integer $n_i \ge 0$ is prescribed, $\sum_i n_i = n$. The problem is to decide whether G admits an $\mathcal{L}$-coloring in which each color i occurs precisely n_i times.

Answering a problem raised by de Werra, recently Dror, Finke, Gravier and Kubiak [51] proved that this problem is NP-complete, already for linear forests and restricting $\mathcal{L}$ to 2-assignments. On the other hand, applying dynamic programming, it is shown that if $|\mathbb{L}| \le p$, where $p \in \mathbb{N}$ is fixed (not part of the input), the problem is solvable in polynomial time on P_n and also on the vertex-disjoint unions of paths. The case of $|\mathbb{L}| = 3$ was solved previously by Xu [198].

It remains open to investigate the complexity of the problem on *trees*, with a bounded total number of colors.

4.4 Choosability

While the hard instances of LIST COLORING turn out to be NP-complete, with respect to CHOOSABILITY the class Π_2^p plays the role of NP. The 2-choosable graphs can be recognized in linear time, by the structural characterization (Theorem 3.3). Apart from this 'smallest' case, essentially every other class of instances is provably hard. The following result gives a complete answer to the problem formulated at the beginning of this section.

Theorem 4.9. (Gutner, Tarsi [83]) *For every $k \geq 3$, k-CHOOSABILITY is Π_2^p-complete.*

The first complexity result of this kind was due to by Rubin [62], but not for lists of equal length. Recently, Gutner proved several similar theorems. In order to state some of them, we need to introduce the following concept.

(2,3)-CHOOSABILITY ((2,3)-CH):

Instance: Graph $G = (V, E)$, number $\ell_i \in \{2, 3\}$ for each vertex v_i.

Question: Does G have a list coloring for every assignment $\mathcal{L} = (L_1, \ldots, L_n)$ such that $|L_i| = \ell_i$ for all $1 \leq i \leq n$?

Along these lines, a large class of problems parametrized by sets S of natural numbers can also be defined, assuming that $\ell_i \in S$ and $|L_i| = \ell_i$ for each vertex v_i in the list assignment $\mathcal{L}$ for which the colorability has to be tested. With this formulation, Rubin's theorem states that (2,3)-CH *is Π_2^p-complete on bipartite graphs.* Gutner proves the following stronger related results.

Theorem 4.10. (Gutner [82]) *Each of the following problems is Π_2^p-complete:*

(2,3)-CH	*on planar bipartite graphs,*
3-CH	*on planar triangle-free graphs,*
4-CH	*on planar graphs,*
3-CH	*on the union of two forests.*

These results may raise the impression that choosability is always at least as hard as list colorability. This is not at all the case, however, as shown by the comparison of the next result with Theorem 4.7. We denote by (p, q, r)-CH the choice version of (p, q, r)-LC.

Theorem 4.11. *(Kratochvíl, Tuza, Voigt [137]) If $2r \geq p$ and $4q > 3r+p$, and also if $2r \leq p$ and $4q > 2p+r$, then the (p, q, r)-CH problem is solvable in linear time.*

The complexity of (p, q, r)-CH, however, is not known in general.

4.5 Graph coloring games

Several games on graphs may be viewed as on-line versions of precoloring extension: At each step, the next player has to extend the partial coloring to a larger one. Here we consider some two-person games of this flavor.

In the variants below, it will be assumed throughout that, already at the beginning of the game, both players know the entire graph $G = (V, E)$ to be colored. Moreover, a color bound k is given. A legal move consists of choosing a vertex v not colored so far, and assign to it an arbitrary color $i \in \{1, \ldots, k\}$ that has not been assigned to any neighbor of v. We begin with a framework that may be viewed as most general in some sense, and then discuss some particular cases and variants.

Achievement and Avoidance Games. In both types, the players move alternately, and the player to move next is obliged to color a vertex, whenever the partial coloring admits an extension. The Achievement Game is won by the player who makes the last legal move; while in the Avoidance Game, the last-but-one move wins, i.e., the winner is the player who can force the other one to make the last move.

Small values of k lead to some concepts interesting on their own: For $k = 2$ both games end up with an inclusionwise maximal bipartite induced subgraph (with *unchangeable* vertex 2-coloring in each of its components), and for $k = 1$ they result in a nonextendable independent set.

These games have been considered by Harary and Tuza [92] for some rather restricted types of graphs G (paths, cycles, Petersen graph) with color bound $k = \chi(G)$. As may be expected, Avoidance turns out to be more complicated than Achievement. Very little is known so far in general, however, though it would be interesting to see various winning strategies, as well as arguments showing that it is hard to determine the winner already on some graph classes of a fairly transparent structure.

For small k, the game is known to be PSPACE-complete on unrestricted graphs, by the results of Schaefer ($k = 1$) and Bodlaender ($k = 2$).

Theorem 4.12. *([158], [20]) For color bounds $k = 1$ and $k = 2$, it is PSPACE-complete to decide who has a winning strategy in the Achievement Game.*

So far, the case of $k \geq 3$ colors seems to be open. On the other hand, more results are available under the condition that the players have to color the vertices in a *prescribed order*. See Bodlaender [20] and Bodlaender and Kratsch [23] for details on those 'sequential coloring' games.

Symmetric strategies. The simplest example to illustrate the idea how the symmetry of a graph can be used successfully, is the winning strategy of the first player in the Achievement Game on the path P_n, n odd. Denoting $P_n = v_0 v_1 \cdots v_{2t}$, Player 1 colors the middle vertex v_t first (with any color), and then 'reflects' each move of

Player 2 to v_t ; i.e., if Player 2 colors some v_i with color j, then Player 1 assigns the same color j to the vertex v_{2t-i} in the next move.

In his recent work, Arroyo [13] applies this idea and its modifications in designing winning strategies for the Achievement and/or Avoidance Games on various types of graphs. Moreover, he considers several further variants of these games, e.g., where each player has to use a prescribed set of colors (those sets may be disjoint for the two players), or adjacent vertices must get the *same* color, etc.

Achievement for $k = 1$ (Node Kayles). The game with one color seems to be of major importance, because the case of more colors can be reduced to it. Indeed, as Arroyo observes [13], the winner is the same on G with k colors and on the Cartesian product $G \Box K_k$ with one color. (The vertex set of $G \Box K_k$ is $V(G) \times \{1, \ldots, k\}$, and two of its vertices (v, i) and (v', i') are adjacent if and only if $i = i'$ and $vv' \in E(G)$ or $i \neq i'$ and $v = v'$.)

If just one color is available, the players sequentially construct larger and larger independent sets until a maximal one is reached, and the first player wins if and only if the set eventually obtained has odd cardinality. Beside the complexity result mentioned above, Schaefer [158] proves that the bipartite version of the game is PSPACE-complete, as well, i.e., where G is supposed to be bipartite, say with vertex partition $V = V_1 \cup V_2$, and player i $(i = 1, 2)$ selects a vertex of V_i in each step.

Finbow and Hartnell [66] investigate, under which conditions is the outcome of the game independent of the actual strategies of the players, i.e., when are the maximal independent sets of G all of the same parity. They prove that for graphs of girth at least 8, the necessary and sufficient condition is that every vertex of degree greater than 1 is adjacent to an odd number of pendant vertices. (The girth condition cannot be weakened here, as shown by the cycle C_7.)

The Achievement Game with $k = 1$ on *paths* is discussed by Berlekamp, Conway and Guy [16, pp. 88–90] in a different but equivalent form, under the name ‘Dawson's Chess’ (played on a $3 \times n$ board with n white pawns and n black pawns, initially placed in the first and third row, respectively; capture is obligatory). Interestingly enough, the score turns out to be ultimately periodic modulo 34. The *second* player has a winning strategy on P_n *if and only if* $n \equiv 4, 8, 20, 24, 28 \pmod{34}$ or $n = 14$ or $n = 34$.

Game chromatic number. This interesting concept was introduced by Bodlaender [20]. Depending on the parity of $n = |V(G)|$, the game becomes some kind of Achievement (n odd) or avoidance (n even), but now the first player wins if and only if the entire graph gets colored. The *game chromatic number* of G, denoted $\chi_g(G)$, is the smallest integer k such that the first player wins the game with color bound k. (In order to avoid some anomalies, Kierstead et al. propose a slight change in the rules, namely that Player 2 begins but he is allowed to pass.)

Faigle, Kern, Kierstead and Trotter [64] proved $\chi_g(T) \leq 4$ for every tree T, and Bodlaender [20] showed that this estimate is tight, by constructing a tree with $\chi_g = 4$. (Let T be the caterpillar with 4 internal nodes along a path, each of degree 4.) The upper bound has been generalized by Kierstead and Tuza [127] who proved that

$$\chi_g(G) \leq 6\, tw(G) - 2$$

holds for every graph G, where $tw(G)$ denotes the treewidth of G (see the definition before Theorem 4.8). It is not known, however, whether the coefficient 6 is really necessary here, or it can be replaced by a smaller one (with possibly a worse error term).

It was conjectured by Bodlaender [20] and proved by Kierstead and Trotter [126] that the game chromatic number of planar graphs is bounded above by a constant. The largest possible value of χ_g, however, is known neither for planar graphs (it is between 8 and 33), nor for outerplanar graphs (between 6 and 8). For a general upper bound, we recall the following result.

Theorem 4.13. *(Kierstead, Trotter [126]) There exists a function $g : \mathbb{N} \to \mathbb{N}$ such that, if a graph G does not contain any subgraph homeomorphic to K_t, then $\chi_g(G) \leq g(t)$.*

It follows, in particular, that the game chromatic number is bounded above by a function of the genus.

Though there is relatively little known about the behavior of the game chromatic number so far, it seems to offer a promising area for research, certainly with a lot more to discover.

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