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Listing nilpotent Lie algebras

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LieAlgDB – a database of Lie algebras

Millennium project: Classification of groups of order at most 2000 by Eick et al. Available in GAP and MAGMA.

LieAlgDB: Similar project to classify some small-dimensional Lie algebras.

Major players: B. Eick, W. de Graaf, M. Costantini.

Classification of soluble and nilpotent Lie algebras

Wilkinson ('88): groups with order p^7 and exponent p .

Ancochea-Bermudez and Goze ('89): Lie algebras of dim at most 7 over \mathbb{C} and \mathbb{R} .

O'Brien ('90): the p -group generation algorithm.

Patera and Zassenhaus ('90): soluble Lie algebras of dimension at most 4 over perfect fields.

Gong ('97): nilpotent Lie algebras of dimension 7 over alg closed fields and \mathbb{R} .

de Graaf ('05): soluble Lie algebras of dimension at most 4.

Newman, O'Brien, Vaughan-Lee: nilpotent Lie rings with order p^7 .

A computational approach to the classification

If L is a nilpotent Lie algebra, then

$$L > L' = \gamma_2(L) > \gamma_3(L) > \cdots > \gamma_c(L) > \gamma_{c+1}(L) = 0.$$

L is an immediate descendant of $L/\gamma_c(L)$.

Stepsize: $\dim \gamma_c(L)$. ■

Immediate descendant algorithm

Input: A nilpotent \mathbb{F}_q -Lie algebra L of class $c - 1$.

Output: The set of \mathbb{F}_q -Lie algebras K of class c such that $K/\gamma_c(K) \cong L$.

The descendant algorithm

An example

a

b

$$[a, b] = c$$

$$[a, c] = x_1$$

$$[b, c] = d$$

$$[a, d] = x_2$$

$$[b, d] = x_3$$

$$[x_1, x] = 0$$

$$[x_2, x] = 0$$

$$[x_3, x] = 0$$

Example. Let

$$L = \langle a, b, c, d \mid [a, b] = c, [b, c] = d, \text{ other products zero} \rangle. \blacksquare$$

We construct the **Lie cover** L^* of L .

L^* contains all immediate descendants of L as a quotient. \blacksquare

We complete the multiplication table of L with new central generators.

- c and d are defined with $[a, b] = c$ and $[b, c] = d$; not touched.
- change products: $[a, c] = 0 + x_1$, $[a, d] = 0 + x_2$, $[b, d] = 0 + x_3$.
- $[c, d] \in [\gamma_2(L), \gamma_3(L)] = \gamma_5(L)$, so $[c, d] = 0$ in L^* .
- The new generators are central: $[x_i, *] = 0$.

The new algebra

We obtained

$$\overline{L^*} = \langle a, b, c, d, x_1, x_2, x_3 \mid [a, b] = c, [b, c] = d, [a, c] = x_1, [a, d] = x_2, [b, d] = x_3 \rangle. \blacksquare$$

Problem: $\overline{L^*}$ is not a Lie algebra.

We have to check Jacobi identities:

$$[a, b, c] + [b, c, a] + [c, a, b] = [c, c] + [d, a] - [x_1, b] = -x_2.$$

Thus $x_2 = 0$. \blacksquare

Hence we obtain L^* :

$$L^* = \langle a, b, c, d, x_1, x_3 \mid [a, b] = c, [b, c] = d, [a, c] = x_1, [a, d] = 0, [b, d] = x_3 \rangle.$$

Finding immediate descendants

Lie algebra:

$$L = \langle a, b, c, d \mid [a, b] = c, [b, c] = d, \text{ other products zero} \rangle .$$

Lie cover:

$$L^* = \langle a, b, c, d, x_1, x_3 \mid [a, b] = c, [b, c] = d, [a, c] = x_1, [a, d] = 0, [b, d] = x_3 \rangle . \blacksquare$$

Lie multiplier: subspace $M = \langle x_1, x_3 \rangle$ spanned by new generators. \blacksquare

Lie nucleus: last term of LCS in L^* . $N = \langle x_3 \rangle$. \blacksquare

Theorem: $0 \leq V < M$. L^*/V is an immediate descendant of L iff $V + N = M$.
Such a subspace is allowable.

Example: every 1-dimensional subspace different from $\langle x_3 \rangle$ is allowable.

Immediate descendants of our example

Problem: Two allowable subspaces may give isomorphic algebras. ■

The action of $\text{Aut}(L)$ can be extended to M using $x_1 = [a, c]$, $x_3 = [b, d]$. Hence $\text{Aut}(L)$ permutes the subspaces of M .

Theorem: Two allowable subspaces give isomorphic Lie algebras iff they are in the same $\text{Aut}(L)$ -orbit. ■

Our example over \mathbb{F}_2 : orbit reps are $\langle x_1 \rangle$ and $\langle x_1 + x_3 \rangle$. ■

Two immediate descendants:

$$K_1 = \langle a, b, c, d, x \mid [a, b] = c, [b, c] = d, [b, d] = x, \text{ other products zero} \rangle .$$

and

$$K_2 = \langle a, b, c, d, x \mid [a, b] = c, [b, c] = d, [a, c] = [b, d] = x, \text{ other products zero} \rangle .$$

A generic computation

Suppose that

$$L = \langle 1, 2, 3, 4, 5 \mid [1, 2] = 3, [1, 3] = 4, [1, 4] = 5 \rangle$$

over \mathbb{F}_q .

Multiplicator: $\langle [2, 3] = 6, [1, 5] = 7, [2, 5] = 8, [3, 4] = -8, [3, 5] = [4, 5] = 0 \rangle$;
 nucleus: $\langle 7, 8 \rangle$.

Number of allowable subspaces: $q^2 + q$. Then

$$\text{Aut}(L) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{11}a_{22} & a_{11}a_{23} & a_{11}a_{24} \\ 0 & 0 & 0 & a_{11}^2 a_{22} & a_{11}^2 a_{23} \\ 0 & 0 & 0 & 0 & a_{11}^3 a_{22} \end{pmatrix}.$$

$$|\text{Aut}(L)| = (q - 1)^2 q^7.$$

Orbits and stabilisers

Orbit 1

Representative: $\langle (1, 0, 0), (0, 0, 1) \rangle$

Stabiliser: $S_1 = \text{Aut}(L)$

Orbit size: 1

Orbit 2

Representative: $\langle (1, 0, 0), (0, 1, -1) \rangle$

Stabiliser: $S_2 = \langle a_{12} = a_{22} - a_{11}, a_{24} = (-1/2)a_{23}^2/a_{22} \rangle$

Orbit size: q^2

Orbit 3

Representative: $\langle (1, -1, 0), (0, 0, 1) \rangle$

Stabiliser: $S_3 = \langle a_{22} = a_{11}^3 \rangle$

Orbit size: $q - 1$

The number of points in total is $q^2 + q$.

Computing the Automorphism group

```
gap> F := GF( 97 ); L := NilpotentLieAlgebras( F, 5 )[8];
gap> P := PolynomialRing( F, 26 );
gap> a11 := P.1; SetName( P.1, "a11" );
....
gap> a55 := P.25; SetName( P.25, "a55" );
gap> D := P.26; SetName( P.26, "D" );
gap> m := [[a11,a12,a13,a14,a15],
>         [a21,a22,a23,a24,a25],
>         [a31,a32,a33,a34,a35],
>         [a41,a42,a43,a44,a45],
>         [a51,a52,a53,a54,a55]];
gap> A := PolynomialsOfAutomorphismGroup( L );
gap> AA := Ideal( P, [ a54, a53, a52, a51, a43, a42, a41, a32, a31, a21,
> a11*a44-a55,
> a11*a34-a45,
> a11*a33-a44,
> a11*a24-a35,
> a11*a23-a34,
> a11*a22-a33,
> Determinant( m )*D-One( F )]);
gap> if not EqualIdeals( A, AA ) then
> Error( "Automorphism group is not as expected" );
gap> fi;
```

Computing a stabiliser

```
gap> v := [[0,1,1]]*One( F );
gap> newgens := PolynomialsOfStabilizer( mat, v );
gap> A := Ideal( P, newgens );
gap> AA := Ideal( P, [ a54, a53, a52, a51, a43, a42, a41, a32, a31, a21,
> a11*a44-a55,
> a11*a34-a45,
> a11*a33-a44,
> a11*a24-a35,
> a11*a23-a34,
> a11*a22-a33,
> a11+a12-a22,
> -2*a22*a24+a23^2,
> Determinant( m )*D-One( F )]);

gap> if not EqualIdeals( A, AA ) then
gap>   Error( "Stabiliser of the second step-1 space is not as expected" );
gap> fi;
```

Another instructive example

Compute step-2 descendants of $L = \mathbb{F}_q^4$.

$$\text{Aut}(L) = \text{GL}(4, q).$$

$$\text{Multiplier} = \text{Nucleus} = W = L \wedge L.$$

$$W = \langle e_1 = [1, 2], e_2 = [1, 3], e_3 = [1, 4], f_3 = [2, 3], f_2 = [4, 2], f_3 = [3, 4] \rangle.$$

Orthogonal form on W : $(e_i, e_j) = (f_i, f_j) = 0$; $(e_i, f_j) = \delta_{ij}$.

$\text{Aut}(L)$ preserves form modulo scalars: $(xg, yg) = (\det g)(x, y)$.

The subspaces of W

There are 4 different 4-dimensional subspaces U of W :

- (i) form is non-degenerate on U with type $+$:
 $\langle e_1, e_2, f_1, f_2 \rangle$.
- (ii) form is non-degenerate on U with type $-$:
 $\langle (0, 0, 2, 1, -2, 0), (0, 1, 2, 0, -a, 0), (1, 0, -2, 0, -a, 0), (0, 0, 0, 0, 0, 1) \rangle$
where $a_1/2$ is not a square.
- (iii) form is degenerate on U with 1-dim kernel: $\langle e_1 + f_1, e_2, e_3, f_2 \rangle$.
- (iv) form is degenerate on U with 2-dim kernel: $\langle e_1, e_2, e_3, f_1 \rangle$.

Results so far

dimension	1	2	3	4	5	6	7	8	9
# nilp. \mathbb{F}_2 -Lie algs	1	1	2	3	9	36	202	1831	27073
# nilp. \mathbb{F}_3 -Lie algs	1	1	2	3	9	34	199		
# nilp. \mathbb{F}_5 -Lie algs	1	1	2	3	9	34	211		

Problem: Large number of subspaces for orbit computations.

E.g. compute 8-dimensional Lie algebras with type (5,3). Step-3 immediate descendants of abelian Lie algebra $\langle x_1, \dots, x_5 \rangle$.

$$M = N = \langle [x_i, x_j] \mid i < j \rangle.$$

Hence $\dim M = 10$, and every 3-dim subspace is allowable.

#(allowable subspaces): 6,347,715 (over \mathbb{F}_2), $1.8 \cdot 10^{11}$ (over \mathbb{F}_3), $6.2 \cdot 10^{15}$ (over \mathbb{F}_5).

General results

O'Brien, Newman, Vaughan-Lee (2004): For $p \geq 5$, there are

$3p^2 + 39p + 344 + 24 \gcd(p - 1, 3) + 11 \gcd(p - 1, 4) + 2 \gcd(p - 1, 5)$
nilpotent Lie rings with size p^6 .

O'Brien, Vaughan-Lee (2005) For $p \geq 5$, there are

$$\begin{aligned} & 3p^5 + 12p^4 + 44p^3 + 127p^2 + 707p + 2455 \\ & + (4p^2 + 44p + 291) \gcd(p - 1, 3) + (p^2 + 19p + 13) \gcd(p - 1, 4) \\ & + (3p + 31) \gcd(p - 1, 5) + 4 \gcd(p - 1, 7) \\ & + 5 \gcd(p - 1, 8) + \gcd(p - 1, 9) \end{aligned}$$

nilpotent Lie rings with size p^7 . For $p \geq 5$, the number of 7-dimensional nilpotent Lie algebras over \mathbb{F}_p is

$$174 + 7p + 2 \gcd(p - 1, 3).$$

Implementation

Sophus package: GAP 4 implementation for

1. (i) the immediate descendant algorithm;
2. (ii) an algorithm to compute the automorphism group;
3. (iii) a simple isomorphism testing;

References

C Schneider. A computer-based approach to the classification of nilpotent Lie algebras. *Experiment. Math.* 2005. Also on arXiv.org.

Sophus package and some classifications obtained by the package are available on www.sztaki.hu/~schneider/Research