

# ON THE SOLUBLE LENGTH OF GROUPS WITH PRIME-POWER ORDER

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To B. H. Neumann in his 90th year.

ABSTRACT. We show that for every integer  $k \geq 3$  and every prime  $p \geq 5$  there is a group with soluble length  $k$  and order  $p^{2^k-2}$ .

## 1. INTRODUCTION

There has been interest since the time of Burnside in the question: given a prime  $p$  and a positive integer  $k$  what is the smallest order of a group of  $p$ -power order with soluble length (exactly)  $k$ ? Let  $p^{\beta_p(k)}$  denote this smallest order.

The first paper which discussed problems like this is one by Burnside [4] in 1913. In that paper he observed that there are groups of order  $p^3$  with soluble length 2; and groups of order  $p^6$  with soluble length 3. Moreover he showed that a group with soluble length  $k+1$  must have order at least  $p^{3k}$ ; and said: but it seems probable that for greater values of  $k$  the actual lower limit for the order exceeds  $p^{3k}$ .

Burnside [5] confirmed this expectation by proving:  $p^{\frac{1}{2}(k+1)(k+2)}$  is a lower limit for the order of a prime power group whose  $k$ -th derived group is not the identity; and moreover: this lower limit is not attained except when  $k$  is 1 or 2. So, in particular, a  $p$ -group with soluble length 4 has order at least  $p^{11}$ .

The theme was taken up and a major advance was made by P. Hall and reported in his now famous paper [8] of 1934. In it he proved that  $2^{k-1} + k - 1 \leq \beta_p(k) \leq 2^{k-2}(2^{k-1} - 1)$  (pp. 56-7). For  $k \leq 4$  this lower bound is no better than Burnside's bound. Hall also established that  $\beta_2(k) \leq 2^k - 1$ .

In 1950 Itô [12] refined the upper bound given by Hall. He showed that  $\beta_p(k) \leq 3 \cdot 2^{k-1}$ . Further progress was made by Blackburn in his thesis [1] in 1956. [The relevant part of that work has never been otherwise published or announced. We are indebted to Professor Blackburn for recently supplying us with a copy of his thesis.] Blackburn proved (Theorem 10) for every  $p > 3$  every group of order  $p^{13}$  has soluble length at most 3; therefore  $\beta_p(4) \geq 14$ . He also described examples of Hall, later published in [9], of groups with

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order  $p^{2^k-1}$  and soluble length  $k$  (for  $p$  odd). (Details of the Hall examples can be found conveniently in Huppert [11] as *Satz* III.17.7.)

In this paper we present examples which complete the story for  $k = 4$ . Namely, there are  $p$ -groups with soluble length 4 and order  $p^{14}$  for all  $p \geq 5$ . More generally, in Section 3 we describe a series of finite  $p$ -groups with soluble length  $k$  and order  $p^{2^k-2}$  for all  $p \geq 5$ . In the context of this problem the primes 2, 3 behave differently already for soluble length 3 (see Blackburn [2], pp. 89-91).

One of the contributions that Burnside and Hall made was to show the significance of commutators in the study of groups of prime-power order. Hence some results about  $p$ -groups come as corollaries of results about nilpotent groups. In Section 2 we show that there is a 2-generator torsion-free nilpotent group with soluble length 4 and Hirsch length 14 (the Hirsch length is the number of infinite cyclic factors in a polycyclic series). This group has nilpotency class 11. Factoring out the  $p$ -th powers of the generators gives that for all  $p \geq 11$  there is  $p$ -group with soluble length 4, order  $p^{14}$  and exponent  $p$ . Examples for the primes 5 and 7 are also described in Section 2.

We are indebted to Professor Avinoam Mann for drawing our attention to his unpublished result that  $2^{k-1} + 2k - 4 \leq \beta_p(k)$  and to Blackburn's thesis. We are also indebted to Dr Werner Nickel for his helpful comments on a draft.

While, in some sense, the upper and lower bounds have the same order of magnitude, there is still a gap to fill even at this level. The quoted results give  $k - 1 < \log_2 \beta_p(k) < k$ . Does  $\log_2 \beta_p(k) - k$  have a limit and, if so, what is it?

## 2. SOLUBLE LENGTH 4

Consider the pro- $p$ -presentation

$$\{ a, b \mid a^p = 1, b^p = 1, [b, a, b] = 1, [b, a, a, a, a, a] = 1, [b, a, a, a, b, a, a, b, a, a, a] = 1 \}.$$

With the help of the  $p$ -Quotient Program (Havas et al. [10]) it is easy to establish that the pro- $p$ -group  $G_p$  defined by this presentation is a finite  $p$ -group of order  $p^{14}$  for  $p = 5, 7, 11$  – and other individual primes as far as resources allow. Moreover, using the same program, one can see that the presentation

$$\{ a, b \mid a^p = 1, b^p = 1, [b, a, b] = 1, [b, a, a, a, a, a] = 1, [b, a, a, a, b, a, a, b, a, a, a] = 1, [[[b, a, a, a], [b, a]], [[b, a, a], [b, a]]] = 1 \}$$

defines a finite  $p$ -group of order  $p^{13}$  for the same primes. Thus  $G_p$  must have soluble length at least 4.

A more general result can be obtained by using the Nilpotent Quotient Program (Nickel [14]) as follows. Let  $G$  denote the group defined by the presentation

$$\{ a, b \mid [b, a, b] = 1, [b, a, a, a, a, a] = 1, [b, a, a, a, b, a, a, b, a, a, a] = 1 \}.$$

Then the program shows that the largest class 11 quotient of  $G$  has Hirsch length 14. Moreover the largest class 11 quotient of the group  $H$  defined by the presentation

$$\{ a, b \mid \begin{array}{l} [b, a, b] = 1, [b, a, a, a, a, a] = 1, \\ [b, a, a, a, b, a, a, b, a, a, a] = 1, \\ [[[b, a, a, a], [b, a]], [[b, a, a], [b, a]]] = 1 \end{array} \}$$

has Hirsch length 13.

Hence the largest torsion-free and class 11 quotient of  $G$  has Hirsch length 14 and soluble length 4.

The presentations used in this section were suggested by some in Caranti et al. [6] and the considerations in the next section.

### 3. THE GENERAL CASE

Our starting point is the  $p$ -adic Lie algebra  $\mathcal{T}$  of dimension 8 described in Caranti et al. [6]. Note that  $\mathcal{T}$  is a free  $\mathbb{Z}_p$ -module with  $\mathbb{Z}_p$ -basis  $\{x, y, c, d, v, s, t, w\}$  and the multiplication can be described by the following table:

$$\begin{array}{llllll} [y, x] = c, & [c, x] = d, & [c, y] = 0, & [d, x] = v, & [d, y] = 0, \\ [v, x] = s, & [v, y] = t, & [s, x] = 0, & [s, y] = 2w, & [t, x] = w, \\ & [t, y] = 0, & [w, x] = px, & [w, y] = -2py. \end{array}$$

It is shown in Caranti et al. that the quotients  $\mathcal{T}_j/\mathcal{T}_{j+1}$  of the lower central series have characteristic  $p$ . The dimensions (over the field of  $p$  elements) of the  $\mathcal{T}_j/\mathcal{T}_{j+1}$  are periodic with period 2, 1, 1, 1, 2, 1. The terms  $\mathcal{D}_k$  of the derived series are easy to calculate and this gives that  $\mathcal{T}_{j+1} < \mathcal{D}_k < \mathcal{T}_j$ , for  $k \geq 2$ , where  $j = 3 \cdot 2^{k-1} - 1$  (thus  $\mathcal{T}_j/\mathcal{T}_{j+1}$  is the second 2-dimensional lower central factor of the appropriate period). This means that  $\mathcal{D}_k$  is one of the  $p + 1$  ideals between  $\mathcal{T}_{j+1}$  and  $\mathcal{T}_j$ . Factoring out one of the  $p$  other ideals gives an algebra of class  $j$ , soluble length  $k + 1$  and dimension  $2^{k+1} - 2$ . The same  $p$ -adic algebra appears in Klaas et al. [13] (p. 51) (taking  $\Pi = \sqrt{p/3}$ ). Using the Cayley map  $x \mapsto (1 - x)(1 + x)^{-1}$  whose properties are described on pp. 31–7 gives a pro- $p$ -group with the corresponding lattice of normal subgroups. The corresponding quotient is a group with soluble length  $k + 1$  and order  $p^{2^{k+1}-2}$ .

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