Small derived quotients in finite p-groups

Csaba Schneider

Informatics Research Laboratory
Computer and Automation Research Institute
The Hungarian Academy of Sciences
csaba.schneider@sztaki.hu
www.sztaki.hu/∼schneider

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Outline

The problem and early results

Mann's result

Our contribution

Philip Hall 1934

The order of $|G^{(k)}/G^{(k+1)}|$

Lemma

Suppose $|G| = p^n$*,* $N \lhd G$*, non-abelian,* $N \leqslant \gamma_i(G)$*. Then*

$$|N/N'| \geqslant p^{i+1} \quad \textit{and} \quad |N| \geqslant p^{i+2}.$$

Corollary

If $G^{(d+1)} \neq 1$*, then*

$$\log_p |G^{(d)}/G^{(d+1)}| \geqslant 2^d + 1$$

and

$$\log_p |G| \geqslant 2^{d+1} + d + 1.$$

Definition

Call $G^{(d)}/G^{(d+1)}$ *small* if $G^{(d+1)} \neq 1$ and $\log_p |G^{(d)}/G^{(d+1)}| = 2^d + 1$.

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Some open questions
What are the sharp bounds?

Hall's bound for $|G^{(d)}/G^{(d+1)}|$:

(i) G/G' can be small;
(ii) G'/G'' can also be small; (see [Sch, JA, 2003]).
(iii) We don't know if $G^{(d)}/G^{(d+1)}$ can be small for $d \geqslant 2$ and p odd.

Hall's bound for $|G|$:

(i) Hall: If $G^{(d)} \neq 1$ then $\log_p |G| \geqslant 2^d + d$.
(ii) Smallest examples with $p \geqslant 5$ and $G^{(d)} \neq 1$ have order $2^{d+1} - 2$ (Evans-Riley, Newman, Schneider 1999).

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Small derived quotients
Their structure

Suppose that $G^{(d)}/G^{(d+1)}$ is small. Then

$$G^{(d)} > [G^{(d)}, G] > \cdots > [G^{(d)}, \underbrace{G, \ldots, G}_{2^d \text{ copies}}] \geqslant G^{(d+1)}.$$

Hence $|G^{(d)}/[G^{(d)}, G]| \leqslant p^2$.

Lemma

If $G^{(d)}/[G^{(d)}, G]$ is cyclic, then

$$G^{(d+1)} = [G^{(d)}, G^{(d)}] = [G^{(d)}, [G^{(d)}, G]] \leqslant [G^{(d)}, \underbrace{G, \ldots, G}_{2^d+1 \text{ copies}}].$$

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Small derived quotients
The two types

There are two cases:

Case 1 (diamond type): $|G^{(d)}/[G^{(d)},G]| = p^2$ and

$$G^{(d)} > [G^{(d)},G] > \cdots > [G^{(d)},\underbrace{G,\ldots,G}_{2^d \text{ copies}}] = G^{(d+1)}$$

Case 2 (uniserial type): $|G^{(d)}/[G^{(d)},G]| = p$ and

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Avi Mann 2000

At most two small derived quotient

Theorem

In a finite p-group there are at most two small derived quotients. If so, then there is a d such that

(i) $G^{(d)}/G^{(d+1)}$ *is of diamond type and*

(ii) $G^{(d+1)}/G^{(d+2)}$ *is of uniserial type.*

Corollary

If $G^{(d)} \neq 1$ *then*

$$\log_p |G| \geqslant 2^d + 2d - 2.$$

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Diamond type is in general not possible

Lemma

For odd p, diamond type small derived quotient is only possible for G/G'.

Corollary

Suppose p is odd and G has two small derived quotients. Then

(i) *they must be $|G/G'|$ and $|G'/G''|$;*

(ii) $|G| = p^6$ *and G has class 5.*

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Proof of Lemma I
The filtration

Suppose

(i) p is odd;
(ii) $d \geqslant 1$;
(iii) $G^{(d)}/G^{(d+1)}$ has diamond type;
(iv) wlog $|G^{(d+1)}| = p$.

Define

(i) $N_{2^d} = G^{(d)}$;
(ii) for $i \geqslant 2^d$, $N_{i+1} = [N_i, G]$.
(iii) Now $G^{(d+1)} = N_{2^{d+1}}$.

We have

$$G^{(d)} = N_{2^d} > N_{2^d+1} > \cdots > N_{2^{d+1}-1} > N_{2^{d+1}} = G^{(d+1)}.$$

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Proof of Lemma II

The Lie algebra

Let

$$\overline{L} = \frac{G^{(d-1)}}{[G^{(d-1)}, G](G^{(d-1)})^p} \oplus \bigoplus_{i \geqslant 2^d} \frac{N_i}{N_{i+1}}.$$

Set $L = \langle G^{(d-1)}/([G^{(d-1)}, G](G^{(d-1)})^p) \rangle$.
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Further,

(i) L is degree-1 generated;

(ii) $\dim L_2 = 2$, $\dim L_3 = 1$, $\dim L_4 = 1$;

(iii) $L_4 = L''$.

Lemma

This is not possible for $p \geqslant 3$.

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Summary

- We still don't know how small $G''/G^{(3)}$ can be if $G^{(3)} \neq 1$.
- Using the Lie ring technique I proved that $G^{(d)} \neq 1$, then $\log_p |G| \geqslant 2^d + 3d - 6$.
- We still don't know how small G can REALLY be.