LieAlgDB — A database of Lie algebras

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Millennium project: Classification of groups of order at most 2000 by Eick et al. Available in GAP and MAGMA.

LieAlgDB: Similar package to classify some small-dimensional Lie algebras.

Developed by M. Costantini, W. de Graaf, Sch.

Aims of LieAlgDB:

- Turn existing classifications into computer code.
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Classification of soluble and nilpotent Lie algebras

Wilkinson ('88): groups with order p^7 and exponent p.

Ancochea-Bermudez and Goze ('89): Lie algebras of dim at most 7 over $\mathbb{C}$ and $\mathbb{R}$.

O'Brien ('90): the p-group generation algorithm.

Patera and Zassenhaus ('90): soluble Lie algebras of dimension at most 4 over perfect fields. (Incorrect)

Gong ('97): nilpotent Lie algebras of dimension 7 over alg closed fields and $\mathbb{R}$.

Newman, O'Brien, Vaughan-Lee: nilpotent Lie rings with order p^7.

New classifications

- Nilpotent Lie algebras
  - Sch: Lie algebras with dimension 6 over FF with odd char.
  - Sch: Lie algebras with dimension 7 over $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_5$; dimension 8 and 9 over $\mathbb{F}_2$;
  - de Graaf: Lie algebras with dimension 6 char $\neq 2$.
- Solvable Lie algebras
  - de Graaf: Solvable Lie algebras with dimension at most 4.
- Non-solvable Lie algebras
  - Strade: Non-solvable Lie algebras with dimension at most 6 over FF.

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The classification of nilpotent Lie algebras

Following the p-group generation algorithm (Newman, O'Brien et al.):

If L is a nilpotent Lie algebra, then

$$L > L' = \gamma_2(L) > \gamma_3(L) > \cdots > \gamma_c(L) > \gamma_{c+1}(L) = 0.$$

L is an immediate descendant of $L/\gamma_c(L)$.

Stepsize: $\dim \gamma_c(L)$.

Immediate descendant algorithm

Input: A nilpotent $\mathbf{F}_q$-Lie algebra L of class $c-1$.

Output: The set of $\mathbf{F}_q$-Lie algebras K of class c such that $K/\gamma_c(K) \cong L$.

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Results by Lie algebra generation

dimension	1	2	3	4	5	6	7	8	9
# nilp. $\mathbb{F}_2$-Lie algs	1	1	2	3	9	36	202	1831	27073
# nilp. $\mathbb{F}_3$-Lie algs	1	1	2	3	9	34	199		
# nilp. $\mathbb{F}_5$-Lie algs	1	1	2	3	9	34	211		

Problem: Large number of subspaces for orbit computations.

Theorem (Sch-de Graaf)

If q is odd, then there are 34 isomorphism types of nilpotent Lie algebras over $\mathbb{F}_q$.

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Willem's method

Task: classify a family of Lie algebras.

- Find all possible isomorphism types in the family.
- Use a method based on Gröbner bases to eliminate duplicates.

Willem obtained

- A classification of solvable Lie algebras with dimension 3.
- A classification of solvable Lie algebras with dimension 4.
- A classification of nilpotent Lie algebras with dimension 6 (char $\mathbb{F} \neq 2$).

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The classification of soluble Lie algebras of dimension at most 4

Theorem (de Graaf '05)

The number of soluble Lie algebras of dimension 3 over $\mathbb{F}_q$ is $q+5$ if char $\mathbb{F}_q \neq 2$ and $q+4$ otherwise.
The number of soluble Lie algebras of dimension 4 over $\mathbb{F}_q$ is

$$q^2+3q+9+\begin{cases} 5 & \text{if } q \equiv 1 \pmod 6 \\ 2 & \text{if } q \equiv 2 \pmod 6 \\ 3 & \text{if } q \equiv 3 \pmod 6 \\ 4 & \text{if } q \equiv 4 \pmod 6 \\ 3 & \text{if } q \equiv 5 \pmod 6. \end{cases}$$

"Which is slightly more than the number found in Patera & Zassenhaus."

Nilpotent Lie algebras with dimension 6

Theorem

Let char $\mathbb{F} \neq 2$. Then there are $26 + 4|F^*/(F^*)^2|$ isomorphism types of nilpotent Lie algebras with dimension 6.

Non-solvable Lie algebras

Based on the classification of simple Lie algebras, Strade classified non-solvable Lie algebras with dimension at most 6.

The LieAlgDB package

Is under construction.

- All nilpotent Lie algebras with dimension up to 5.
- All nilpotent Lie algebras of dimension 6 over char $\neq 2$.
- Nilpotent Lie algebras of dimension 7 over $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_5$, and of dimension 8 and 9 over $\mathbb{F}_2$.
- All solvable Lie algebras of dimension at most 4.
- All non-solvable Lie algebras with dimension at most 6 over FF.

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