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## Permutation groups and Cartesian decompositions

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Joint research with Cheryl Praeger and Robert Baddeley

## The big picture

**Problem:** given  $G \leq \text{Sym } \Omega$ , describe the groups  $H$  s.t.  $G \leq H \leq \text{Sym } \Omega$ .

Such an  $H$  is called an **overgroup** of  $G$ , and  $G \leq H$  is called an **embedding**.

**Why?** Think of Cayley graphs and other combinatorial structures.

**Easier problem (still quite difficult):** given a  $\text{Prop}_1$  group  $G \leq \text{Sym } \Omega$ , find all its  $\text{Prop}_2$  overgroups.

$\text{Prop}_1, \text{Prop}_2 \in \{\text{“primitive”}, \text{“quasiprimitive”}, \text{“innately transitive”}\}$ .

## What is known?

Groups of interest:

- **Primitive groups:** No non-trivial invariant partition.
- **QuasiPrimitive groups:** Every non-trivial normal subgroup is transitive.
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- **Praeger (1990):** overgroups of primitive groups ( $\text{Prim} \leq \text{Prim}$ );
  - **Baddeley & Praeger:**  $\text{QP} \leq \text{Prim}$  when  $\text{Prim}$  is not PA or AS;
  - **Praeger:**  $\text{QP} \leq \text{QP}$  embeddings when  $\text{QP}$  is not AS or PA;
  - **Baddeley, Praeger & Schneider:**  $\text{IT} \leq \text{Prim}$  when  $\text{Prim}$  is PA;
  - **Schneider:**  $\text{IT} \leq \text{IT}$ .

## Why Cartesian decompositions?

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**Prototype:** Let  $H \leq \text{Sym } \Gamma$  be primitive, non-regular, and  $K \leq S_\ell$  transitive. Then  $H \text{ wr } K = (H \times \cdots \times H) \rtimes K$  is primitive in product action on  $\Gamma^\ell = \Gamma \times \cdots \times \Gamma$  via:

$$(\gamma_1, \dots, \gamma_\ell)^{(h_1, \dots, h_\ell)} = \left( \gamma_1^{h_1}, \dots, \gamma_\ell^{h_\ell} \right), \quad (\gamma_1, \dots, \gamma_\ell)^k = \left( \gamma_{1k-1}, \dots, \gamma_{\ell k-1} \right).$$

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If  $G \leq H \text{ wr } K$  then  $G$  acts on a Cartesian product  $\Gamma \times \cdots \times \Gamma$  preserving the product structure.

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If  $G \leq H \text{ wr } K$  then  $G$  acts on a Cartesian product  $\Gamma \times \cdots \times \Gamma$  preserving the product structure.

Conversely if  $G \leq \text{Sym } \Omega$  preserves a product structure  $\Gamma \times \cdots \times \Gamma$  then  $G \leq \text{Sym } \Gamma \text{ wr } S_\ell$ .

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### Questions:

- How to find “all ways”?
- How to check if one such decomposition is  $G$ -invariant?

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R2	4	5	6
R3	7	8	9

Bijection  $\Omega \rightarrow \{R1, R2, R3\} \times \{C1, C2, C3\}$ .

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CD is given by  $\{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}\}$  and  $\{\{1, 4, 7\}, \{2, 5, 8\}, \{3, 6, 9\}\}$ .

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Important property:

$$|R_i \cap C_j| = 1. \quad (1)$$

Conversely, if  $\{R_i\}_i$  and  $\{C_j\}_j$  have (1) then there is a bijection  $\Omega \rightarrow \{R_i\}_i \times \{C_j\}_j$ .

## Hence the definition

**Definition** Let  $\Omega$  be a set. A set of partitions  $\Gamma_1, \dots, \Gamma_\ell$  s.t.

$$|\gamma_1 \cap \dots \cap \gamma_\ell| = 1 \quad \text{for all } \gamma_1 \in \Gamma_1, \dots, \gamma_\ell \in \Gamma_\ell$$

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$g \in \text{Sym } \Omega$  **preserves** a CD if it permutes the partitions  $\Gamma_1, \dots, \Gamma_\ell$ .

$G \leq \text{Sym } \Omega$  **preserves** a CD if its elements preserve the CD.

If  $|\Gamma_i| = |\Gamma_j|$  then  $\text{Stab}_{\text{Sym } \Omega}(\text{CD}) \cong \text{Sym } \Gamma_1 \text{ wr } S_\ell$ .

## Cartesian decompositions and innately transitive groups

Let  $G \leq \text{Sym } \Omega$  be an innately transitive group with plinth  $M$ .

**Theorem** If  $\Gamma_1, \dots, \Gamma_\ell$  is a  $G$ -invariant CD then the  $\Gamma_i$  are  $M$ -invariant partitions.

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Fix  $\omega \in \Omega$ , and let  $\gamma_i \in \Gamma_i$  such that  $\omega \in \gamma_i$ . Set  $K_i = M\gamma_i$ . Then

$$\{K_i\} \text{ is } G_\omega\text{-invariant, } \bigcap K_i = M_\omega \quad \text{and} \quad K_i \left( \bigcap_{j \neq i} K_j \right) = M. \quad (2)$$

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**Theorem** For a fixed  $\omega$  there is a one-to-one correspondence between  $G$ -invariant CDs and Cartesian systems in  $M$ .

## Simple plinth: The problem

**Problem:** Let  $G \leq \text{Sym } \Omega$  be IT, with a non-abelian simple plinth  $T$ . Find all embeddings  $G \leq \text{Sym } \Gamma \text{ wr } S_\ell$ .

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**Reduction I:** Find all  $G$ -invariant Cartesian decompositions of  $\Omega$  s.t.  $|\Gamma_i| = |\Gamma_j|$ .

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Cartesian system: subgroups  $K_1, \dots, K_\ell$  such that

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**Baddeley & Praeger (1998, CFSG):**  $\ell = 2$  and the possibilities for  $T, K_1, K_2$  are known.

## Simple plinth: The result

	$G$	$T$	Sym $\Gamma$ wr $S_\ell$	$K_1, K_2$	$ \Omega $
1		$A_6$		$A_5$	
2		$M_{12}$		$M_{11}$	
3		$P\Omega_8^+(q)$		$\Omega_7(q)$	
4		$Sp_4(q)$		$Sp_2(q^2).2$	

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1	$A_6$	$S_6 \text{ wr } S_2$	$A_5$	36
2	$M_{12}$	$S_{12} \text{ wr } S_2$	$M_{11}$	144
3	$P\Omega_8^+(q)$	$S_{ P\Omega_8^+(q):\Omega_7(q) } \text{ wr } S_2$	$\Omega_7(q)$	$ P\Omega_8^+(q) : G_2(q) $
4	$\text{Sp}_4(q)$ $q \geq 4$ even	$S_{q^2(q^2-1)} \text{ wr } S_2$	$\text{Sp}_2(q^2).2$	$q^4(q^2 - 1)^2$

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**Theorem** :  $C_{\text{Sym } \Omega}(T) = 1$ , hence  $T \leq G \leq \text{Aut}(T)$ .

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	$G$	$T$	$\text{Sym } \Gamma \text{ wr } S_\ell$	$K_1, K_2$	$ \Omega $
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2	$M_{12} \leq G \leq \text{Aut}(M_{12})$	$M_{12}$	$S_{12} \text{ wr } S_2$	$M_{11}$	144
3	$\text{P}\Omega_8^+(q) \leq G \leq \text{P}\Omega_8^+(q)\Phi \langle \vartheta \rangle$ $\Phi$ : field automorphisms $\vartheta$ swaps $K_1, K_2$	$\text{P}\Omega_8^+(q)$	$S_{ \text{P}\Omega_8^+(q):\Omega_7(q) } \text{ wr } S_2$	$\Omega_7(q)$	$ \text{P}\Omega_8^+(q) : G_2(q) $
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## Future work

The following projects are in progress.

- Describe Cartesian decompositions preserved by IT groups with non-simple plinth (with Baddeley and Praeger).
- Describe  $IT \leq IT$  inclusions.