Lie algebras, their classification and applications
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Listing nilpotent Lie algebras

Csaba Schneider

Computer and Automation Research Institute
The Hungarian Academy of Sciences

csaba.schneider@sztaki.hu
www.sztaki.hu/∼schneider

Nilpotent Lie algebras

Lower central series:

$$\gamma_i(L) = \langle \text{products of length } i \rangle_{\mathbb{F}} = \langle [x_1, \ldots, x_i] \rangle_{\mathbb{F}}.$$

We have

$$L = \gamma_1(L) \geqslant L' = \gamma_2(L) \geqslant \gamma_3(L) \geqslant \cdots \geqslant \gamma_i(L) \geqslant \cdots$$

L is nilpotent if $\gamma_{c+1}(L) = 1$ for some c. The smallest c is the nilpotency class.

If L is nilpotent then

$$L > L' = \gamma_2(L) > \gamma_3(L) > \cdots > \gamma_c(L) > \gamma_{c+1}(L) = 0.$$

Example

4-dimensional example:

$$L = \langle a,\ b,\ c,\ d \mid [a,b] = c,\ [b,c] = d,\ \text{other products zero} \rangle\ .$$

This is enough to define multiplication in L (distributive).

In L we have $[x_1, x_2, x_3, x_4] = 0$ for all $x_i \in L$.

L is nilpotent with class 3:

$$\gamma_1(L) = L = \langle a,\ b,\ c,\ d \rangle\ ,\ \gamma_2(L) = \langle c,\ d \rangle\ ,\ \gamma_3(L) = \langle d \rangle\ ,\ \gamma_4(L) = 0.$$

Type of L is $(2, 1, 1)$.

$\langle a,\ b,\ c,\ d \rangle$ is a nilpotent basis.

Classification of nilpotent Lie algebras

Brahana (50's): some class-2 p-groups.

Gauger ('73, TAMS): 2-step nilpotent Lie algebras of type $(d, 2)$ over algebraically closed field.

Wilkinson ('88): groups with order p^7 and exponent p.

Romdhani and Mustapha ('89 LMA): nilpotent Lie algebras of dim 7 over $\mathbb{C}$ and $\mathbb{R}$.

Ancochea-Bermudez and Goze ('89): the same.

O'Brien ('90): the p-group generation algorithm.

Tsagas ('99): nilpotent Lie algebras of dim 8 over char 0.

Vaughan-Lee: nilpotent Lie rings with order p^7.

A computational approach to the classification

If L is a nilpotent Lie algebra, then

$$L > L' = \gamma_2(L) > \gamma_3(L) > \cdots > \gamma_c(L) > \gamma_{c+1}(L) = 0.$$

L is an immediate descendant of $L/\gamma_c(L)$.

Stepsize: $\dim \gamma_{c+1}(L)$.

Immediate descendant algorithm

Input: A nilpotent $\mathbf{F}_q$-Lie algebra L of class $c-1$.

Output: The set of $\mathbf{F}_q$-Lie algebras K of class c such that $K/\gamma_c(K) \cong L$.

Classification based on the Immediate Descendant Algorithm

Example: Lie algebras with dimension 5.

Step 1. 1-dimensional Lie algebras, 2-dimensional Lie algebras (abelian).

Step 2. 3-dimensional Lie algebras:

- L is abelian (3).
- L is 1-step immediate descendant of a 2-dimensional (2,1).

Step 3. 4-dimensional Lie algebras:

- L is abelian (4).
- L is 1-step immediate descendant of a 3-dimensional (3,1), (2,1,1).
- L is 2-step immediate descendant of a 2-dimensional (not possible).

Step 4. 5-dimensional Lie algebras:

- L is abelian (5).
- L is 1-step immediate descendant of a 4-dimensional (4,1), (3,1,1), (2,1,1,1)
- L is 2-step immediate descendant of a 3-dimensional (3,2), (2,1,2).
- L is 3-step immediate descendant of a 2-dimensional (not possible).

The descendant algorithm

An example

a b

$[a, b] = c$

$[a, c] = x_1$ $[b, c] = d$

$[a, d] = x_2$ $[b, d] = x_3$

$[x_1, x] = 0$ $[x_2, x] = 0$ $[x_3, x] = 0$

Example. Let

$$L = \langle a,\ b,\ c,\ d \mid [a,b] = c,\ [b,c] = d,\ \text{other products zero}\rangle .$$

We construct the Lie cover L^* of L.

L^* contains all immediate descendants of L as a quotient.

We complete the multiplication table of L with new central generators.

- c and d are defined with $[a,b] = c$ and $[b,c] = d$; not touched.
- change products: $[a,c] = 0 + x_1$, $[a,d] = 0 + x_2$, $[b,d] = 0 + x_3$.
- $[c,d] \in [\gamma_2(L), \gamma_3(L)] = \gamma_5(L)$, so $[c,d] = 0$ in L^*.
- The new generators are central: $[x_i, *] = 0$.

The new algebra

We obtained

$$\begin{aligned}\overline{L^*} &= \langle a,\ b,\ c,\ d,\ x_1,\ x_2,\ x_3 \mid \\ &\quad [a,b]=c,\ [b,c]=d,\ [a,c]=x_1,\ [a,d]=x_2,\ [b,d]=x_3\rangle\,.\end{aligned}$$

Problem: $\overline{L^*}$ is not a Lie algebra.

We have to check Jacobi identities:

$$[a,b,c]+[b,c,a]+[c,a,b]=[c,c]+[d,a]-[x_1,b]=-x_2.$$

Thus $x_2 = 0$.

Hence we obtain L^*:

$$\begin{aligned}L^* &= \langle a,\ b,\ c,\ d,\ x_1,\ x_3 \mid \\ &\quad [a,b]=c,\ [b,c]=d,\ [a,c]=x_1,\ [a,d]=0,\ [b,d]=x_3\rangle\,.\end{aligned}$$

Finding immediate descendants

Lie algebra:

$$L = \langle a,\ b,\ c,\ d \mid [a,b] = c,\ [b,c] = d,\ \text{other products zero} \rangle\ .$$

Lie cover:

$$\begin{aligned} L^* \quad &= \quad \langle a,\ b,\ c,\ d,\ x_1,\ x_3 \mid \\ & \qquad [a,b] = c,\ [b,c] = d,\ [a,c] = x_1,\ [a,d] = 0,\ [b,d] = x_3 \rangle\ . \end{aligned}$$

Lie multiplicator: subspace $M = \langle x_1, x_3 \rangle$ spanned by new generators.

Lie nucleus: last term of LCS in L^*. $N = \langle x_3 \rangle$.

Theorem: $0 \neq V \leqslant M$. L^*/M is an immediate descendant of L iff $V + N = M$. Such a subspace is allowable.

Example: every 1-dimensional subspace different from $\langle x_3 \rangle$ is allowable.

Immediate descendants of our example

Problem: Two allowable subspaces may give isomorphic algebras.

The action of Aut(L) can be extended to M using $x_1 = [a,c]$, $x_3 = [b,d]$. Hence Aut(L) permutes the subspaces of M.

Theorem: Two allowable subspaces give isomorphic Lie algebras iff they are in the same Aut(L)-orbit.

Our example over $\mathbb{F}_2$: orbit reps are $\langle x_1 \rangle$ and $\langle x_1 + x_3 \rangle$.

Two immediate descendants:

$$K_1 = \langle a,\ b,\ c,\ d,\ x \mid [a,b] = c,\ [b,c] = d,\ [b,d] = x,\ \text{other products zero} \rangle\ .$$

and

$$K_2 = \langle a,\ b,\ c,\ d,\ x \mid [a,b] = c,\ [b,c] = d,\ [a,c] = [b,c] = x,\ \text{other products zero} \rangle\ .$$

Results so far

dimension	1	2	3	4	5	6	7	8	9
# nilp. $\mathbf{F}_2$-Lie algs	1	1	2	3	9	36	202	1836	26711
# nilp. $\mathbf{F}_3$-Lie algs	1	1	2	3	9	34	200	?	?

Problem: Large number of subspaces for orbit computations.

E.g. compute 8-dimensional Lie algebras with type (5,3). Step-3 immediate descendants of abelian Lie algebra $\langle x_1, \ldots, x_5 \rangle$.

$M = N = \left\langle [x_i, x_j] \mid i < j \right\rangle$.

Hence $\dim M = 10$, and every 3-dim subspace is allowable.

#(allowable subspaces): 6,347,715 (over $\mathbf{F}_2$), $1.8 \cdot 10^{11}$ (over $\mathbf{F}_3$), $6.2 \cdot 10^{15}$ (over $\mathbf{F}_5$).

One neat trick

Task: Find all type-$(7, 2)$ $\mathbb{F}_2$-Lie algebras. Need to determine the orbits of $\mathrm{GL}(7, 2)$ on the set of 2 dimensional subspaces acting on $\mathbb{F}_2^7 \wedge \mathbb{F}_2^7 \cong \mathbb{F}_2^{21}$.

There are $733, 006, 703, 275$ subspaces.

The number of orbits can be found using the Lemma That Is Not Of Burnside's (aka Cauchy-Frobenius Lemma):

$$\#\text{orbits} = \frac{1}{|G|} \sum_{g \in G} \text{fix}\, g.$$

The number of orbits is 20.

Using the list of groups with order 2^9 we can find 20 Lie algebras.

Implementation

Sophus package: GAP 4 implementation for

1. (i) the immediate descendant algorithm;

2. (ii) an algorithm to compute the automorphism group;

3. (iii) a simple isomorphism testing;

Publicly available soon.