FACTORISATIONS OF CHARACTERISTICALLY SIMPLE GROUPS

CHERYL E. PRAEGER AND CSABA SCHNEIDER

Abstract. We solve some factorisation problems for finite characteristically simple groups, using factorisations of elementary abelian 2-groups. The problems arose from a study of permutation groups. We found the connection between the factorisations of these two types of groups surprisingly interesting.

1. Introduction

A group factorisation is a pair $(G, \{A, B\})$ where G is a group and A, B are subgroups of G such that $AB = G$. It is called non-trivial if both A and B are different from G. In this situation we also say that $\{A, B\}$ is a factorisation of G.

The aim of this paper is to characterise three different kinds of factorisations of finite characteristically simple groups. These factorisations arose in the study [3] of Cartesian decompositions invariant under a permutation group G with a minimal normal subgroup M that is non-abelian and transitive. In our research we found that describing certain types of Cartesian decompositions was equivalent to finding the factorisations of the characteristically simple group M that are studied in this paper.

This work stemmed from a study of finite primitive and quasiprimitive permutation groups [1, 8], in which the problem of classifying all factorisations $(T, \{A, B\})$ of finite simple groups T such that $|T|$, $|A|$, and $|B|$ are divisible by the same primes plays a vital rôle. Such a factorisation is called a full factorisation of the simple group T and their classification was achieved by Baddeley and Praeger [2]. There are surprisingly few examples, and the possible isomorphism types for T, A, and B are listed in Table 1. In each line of Table 1 the conjugacy classes of the groups A and B have to satisfy further

---

Date: 17 October 2001.

2000 *Mathematics Subject Classification.* 20D06, 20D08, 20D40.

Key words and phrases. characteristically simple groups, finite simple groups, factorisations, elementary abelian groups.

The authors acknowledge the support of an Australian Research Council grant.

conditions, and a detailed description of such full factorisations can be found in [2]. In particular, in each line A and B cannot be conjugate.

	T	A	B
1	A_6	A_5	A_5
2	M_{12}	M_{11}	$\mathsf{M}_{11},\ \mathsf{PSL}_2(11)$
3	$\mathsf{P}\Omega_8^+(q),\ q\geqslant 3$	$\Omega_7(q)$	$\Omega_7(q)$
4	$\mathsf{P}\Omega_8^+(2)$	$\mathsf{Sp}_6(2)$	$\mathsf{A}_7,\ \mathsf{A}_8,\ \mathsf{S}_7,\ \mathsf{S}_8,\ \mathsf{Sp}_6(2),\ \mathbb{Z}_2^6\rtimes\mathsf{A}_7,\ \mathbb{Z}_2^6\rtimes\mathsf{A}_8$
		A_9	$\mathsf{A}_8,\ \mathsf{S}_8,\ \mathsf{Sp}_6(2),\ \mathbb{Z}_2^6\rtimes\mathsf{A}_7,\ \mathbb{Z}_2^6\rtimes\mathsf{A}_8$
5	$\mathsf{Sp}_4(q),\ q\geqslant 4$ even	$\mathsf{Sp}_2(q^2)\cdot 2$	$\mathsf{Sp}_2(q^2)\cdot 2,\ \mathsf{Sp}_2(q^2)$

TABLE 1. Full factorisations $\{A,B\}$ of finite simple groups T

A systematic study [3] of Cartesian decompositions left invariant by a finite quasiprimitive permutation group raised the question of classifying full factorisations of finite non-abelian characteristically simple groups, defined below.

If M is a non-abelian characteristically simple group, then the simple normal subgroups of M are denoted by $T_1,\ldots,T_k$. Then M can be written as the direct product $M=T_1\times\cdots\times T_k$, and if T is a group isomorphic to T_i for $i=1,\ldots,k$, then we also identify M with T^k. For $i=1,\ldots,k$ the i-th projection map $M\to T_i$ is denoted by σ_i.

Definition 1.1. *For a characteristically simple group $M=T_1\times\cdots\times T_k$ and proper subgroups K_1, K_2, $(M,\{K_1,K_2\})$ is said to be a* full factorisation *if $M=K_1K_2$ and $(T_i,\{\sigma_i(K_1),\sigma_i(K_2)\})$ is a full factorisation for all i.*

The fact that all A and B occurring in Table 1 are almost simple or perfect (see Lemma 4.1) is the key to our first theorem, which will be proved in Section 4.

Theorem 1.2. *If $(M,\{K_1,K_2\})$ is a full factorisation of a characteristically simple group $M=T_1\times\cdots\times T_k$, then*

$$\sigma_1(K_i)'\times\cdots\times\sigma_k(K_i)'\leqslant K_i \quad \textit{for} \quad i=1,\ 2. \tag{1}$$

For the groups in rows 1–4 in Table 1 we obtain the following corollary, whose proof is found in Section 4.

Corollary 1.3. *Let* $M = T_1 \times \cdots \times T_k$ *be a characteristically simple group such that the* T_i *are isomorphic to one of the groups* T *in rows* 1–4 *of Table* 1*. Then* $(M, \{K_1, K_2\})$ *is a full factorisation if and only if* $(T_j, \{\sigma_j(K_1), \sigma_j(K_2)\})$ *is a full factorisation for* $j = 1, \dots, k$ *and* (1) *holds.*

Corollary 1.3 does not apply to row 5 of Table 1. The difficulty in row 5 is that for the groups in this row $A'B' \neq T$. Let us suppose that $T \cong \mathsf{Sp}_4(q)$ for some even $q \geqslant 4$, and set $M = T_1 \times \cdots \times T_k = T^k$. Let K_1 and K_2 be subgroups of M such that $(T_i, \{\sigma_i(K_1), \sigma_i(K_2)\})$ is a full factorisation for all i and (1) holds. How can we guarantee that $K_1K_2 = M$? If $\{A, B\}$ are subgroups of T such that A, $B \cong \mathsf{Sp}_2(q^2) \cdot 2$ and $AB = T$, then a result of Baddeley and Praeger [2] implies that $A'B = B'A = T$, but $A'B' \neq T$. By Theorem 1.2, K_i' is isomorphic to $(A')^k$, and $\mathsf{N}_M(K_i) \cong (\mathsf{Sp}_2(q^2) \cdot 2)^k$. So $\mathsf{N}_M(K_i)' = K_i'$, and $\mathsf{N}_M(K_i)/K_i'$ is the unique quotient of $\mathsf{N}_M(K_i)$ that is isomorphic to $k\mathbb{Z}_2$. The following theorem gives two necessary and sufficient conditions for the factorisation $M = K_1K_2$ to hold.

Theorem 1.4. *Let* $M = T_1 \times \cdots \times T_k$ *be a characteristically simple group such that* $T_i \cong \mathsf{Sp}_4(q)$ *with some even* $q \geqslant 4$ *for all* i*. Let* K_1 *and* K_2 *be subgroups of* M *such that* $(T_i, \{\sigma_i(K_1), \sigma_i(K_2)\})$ *is a full factorisation for all* i*, and* (1) *holds. Then the following are equivalent:*

1. $K_1K_2 = M$*;*
2. $\Psi_1(K_1) + \Psi_2(K_2) = k\mathbb{Z}_2$ *for some epimorphisms* $\Psi_i : \mathsf{N}_M(K_i) \to k\mathbb{Z}_2$ $(i = 1,\ 2)$ *such that* Ψ_1 *and* Ψ_2 *agree on* $\mathsf{N}_M(K_1) \cap \mathsf{N}_M(K_2)$*;*
3. $\Psi_1(K_1) + \Psi_2(K_2) = k\mathbb{Z}_2$ *for all epimorphisms* $\Psi_i : \mathsf{N}_M(K_i) \to k\mathbb{Z}_2$ $(i = 1,\ 2)$ *such that* Ψ_1 *and* Ψ_2 *agree on* $\mathsf{N}_M(K_1) \cap \mathsf{N}_M(K_2)$*.*

In Example 4.3 we construct two maps Ψ_1 and Ψ_2 which satisfy the conditions of Theorem 1.4. In Section 3 we prove a general version of Theorem 1.4 as Proposition 3.4, which has applications to direct products of almost simple groups. The proof of Theorem 1.4 will be given in Section 4.

Our description of full factorisations of finite characteristically simple groups enables us to characterise a different kind of factorisation, which was also needed in the study [3] of Cartesian decompositions.

Theorem 1.5. *Let* $M = T_1 \times \cdots \times T_{2k}$ *be a characteristically simple group,* $\varphi_i : T_i \to T_{i+k}$ *an isomorphism for* $i = 1, \ldots, k$*, and set*

$$D = \{(t_1, \ldots, t_k, \varphi_1(t_1), \ldots, \varphi_k(t_k)) \mid t_1 \in T_1, \ldots, t_k \in T_k\}.$$

If K *is a subgroup of* M *such that* $\sigma_i(K) \cong \sigma_j(K) \ncong T_1$ *for all* i *and* j*, then the following hold.*

(a) *If* $DK = M$ *then*

$$(T_i, \{\sigma_i(K), \varphi_i^{-1}(\sigma_{i+k}(K))\}) \text{ is a full factorisation for all } i \text{ and } \prod_{i=1}^{2k} \sigma_i(K)' \leqslant K. \tag{2}$$

(b) *If the* T_i *are isomorphic to a* T *in one of the rows* 1–4 *in Table* 1*, then* $DK = M$ *if and only if* $(T_i, \{\sigma_i(K), \varphi_i^{-1}(\sigma_{i+k}(K))\})$ *is a full factorisation for all* i*, and* $K = \sigma_1(K) \times \cdots \times \sigma_{2k}(K)$*.*

(c) *If* $T_i \cong \mathsf{Sp}_4(q)$ *with some even* $q \geqslant 4$ *for all* i *and* K *satisfies* (2) *then the following are equivalent:*
   (1) $DK = M$*;*
   (2) $\Psi(K) + \Psi(D \cap \mathsf{N}_M(K)) = 2k\mathbb{Z}_2$ *for some epimorphism* $\Psi : \mathsf{N}_M(K) \to 2k\mathbb{Z}_2$*;*
   (3) $\Psi(K) + \Psi(D \cap \mathsf{N}_M(K)) = 2k\mathbb{Z}_2$ *for all epimorphisms* $\Psi : \mathsf{N}_M(K) \to 2k\mathbb{Z}_2$*.*

Theorem 1.5 will be proved in Section 6.

A strong multiple factorisation of a finite almost simple group G is an ordered pair $(G, \{A_1, \ldots, A_k\})$ where $k \geqslant 3$ and $A_1, \ldots, A_k$ are proper subgroups of G not containing $\mathrm{Soc}\, G$ such that $A_i \left(\bigcap_{j \neq i} A_j\right) = G$ for all i. This type of factorisation arose in the study [1] of primitive overgroups of finite quasiprimitive groups. It was proved [2] that all strong multiple factorisations of finite non-abelian almost simple groups consist of $k = 3$ subgroups, and the possible isomorphism types for T, A_1, A_2, and A_3 where T is a non-abelian simple group can be found in Table 2.

In our research [3] we needed to consider strong multiple factorisations of characteristically simple groups, defined as follows.

Definition 1.6. *Let* $M = T_1 \times \cdots \times T_k$ *be a characteristically simple group. For subgroups* K_1, K_2, K_3 *of* M, $(M, \{K_1, K_2, K_3\})$ *is said to be a* strong multiple factorisation *if*

(1) $K_1(K_2 \cap K_3) = K_2(K_1 \cap K_3) = K_3(K_1 \cap K_2) = M$*; and*
(2) $(T_i, \{\sigma_i(K_1), \sigma_i(K_2), \sigma_i(K_3)\})$ *is a strong multiple factorisation for all* i*.*

	T	A_1	A_2	A_3
1	$\mathsf{Sp}_{4a}(2),\ a \geqslant 2$	$\mathsf{Sp}_{2a}(4)\cdot 2$	$\mathsf{O}_{4a}^{-}(2)$	$\mathsf{O}_{4a}^{+}(2)$
2	$\mathsf{P}\Omega_8^{+}(3)$	$\Omega_7(3)$	$\mathbb{Z}_3^6 \rtimes \mathsf{PSL}_4(3)$	$\mathsf{P}\Omega_8^{+}(2)$
3	$\mathsf{Sp}_6(2)$	$\mathsf{G}_2(2)$	$\mathsf{O}_6^{-}(2)$	$\mathsf{O}_6^{+}(2)$
		$\mathsf{G}_2(2)'$	$\mathsf{O}_6^{-}(2)$	$\mathsf{O}_6^{+}(2)$
		$\mathsf{G}_2(2)$	$\mathsf{O}_6^{-}(2)'$	$\mathsf{O}_6^{+}(2)$
		$\mathsf{G}_2(2)$	$\mathsf{O}_6^{-}(2)$	$\mathsf{O}_6^{+}(2)'$

TABLE 2. Strong multiple factorisations $\{A_1, A_2, A_3\}$ of finite simple groups T

For groups as in the previous definition, we also say that $\{K_1, K_2, K_3\}$ is a strong multiple factorisation of M. The following theorem is similar to Theorem 1.2, and it will be proved in Section 5.

Theorem 1.7. *If $M = T_1 \times \cdots \times T_k$ is a characteristically simple group with a strong multiple factorisation $(M, \{K_1, K_2, K_3\})$, then*

$$\sigma_1(K_i)' \times \cdots \times \sigma_k(K_i)' \leqslant K_i \quad \textit{for} \quad i = 1,\ 2,\ 3. \tag{3}$$

For the groups in rows 1–2 of Table 2, we obtain the following corollary; see Section 5 for details.

Corollary 1.8. *Let $M = T_1 \times \cdots \times T_k$ be a characteristically simple group such that the T_i are isomorphic to one of the groups T in rows 1–2 of Table 2. Let K_1, K_2, and K_3 be subgroups of M such that $(T_i, \{\sigma_i(K_1), \sigma_i(K_2), \sigma_i(K_3)\})$ is a strong multiple factorisation for all $i \in \{1, \dots, k\}$. Then $(M, \{K_1, K_2, K_3\})$ is a strong multiple factorisation if and only if $K_i = \sigma_1(K_i) \times \cdots \times \sigma_k(K_i)$ for $i = 1,\ 2,\ 3$.*

The case $T \cong \mathsf{Sp}_6(2)$ remains to be treated, and we use methods similar to those in Theorem 1.4 to complete our description of strong multiple factorisations of characteristically simple groups. Let $T \cong \mathsf{Sp}_6(2)$ and set $M = T_1 \times \cdots \times T_k = T^k$. Let K_1, K_2, and K_3 be subgroups of M such that $(T_i, \{\sigma_i(K_1), \sigma_i(K_2), \sigma_i(K_3)\})$ is a strong multiple factorisation for all i, and (3) holds. Then for each i the factorgroup $\mathsf{N}_M(K_i)/K_i'$ is an elementary abelian 2-group with order 2^k, and is the unique such quotient of $\mathsf{N}_M(K_i)$. The following theorem gives two criteria in terms of the subgroups K_i/K_i' to decide when $(M, \{K_1, K_2, K_3\})$ is a strong multiple factorisation. Its proof will be given in Section 5.

Theorem 1.9. *Let $M = T_1 \times \cdots \times T_k$ where the T_i are isomorphic to $\mathsf{Sp}_6(2)$, and let K_1, K_2, and K_3 be subgroups of M such that $(T_i, \{\sigma_i(K_1), \sigma_i(K_2), \sigma_i(K_3)\})$ is a strong multiple factorisation for all i, and (3) holds. Then the following are equivalent.*

(1) *The factorisations $K_1(K_2 \cap K_3) = K_2(K_1 \cap K_3) = K_3(K_1 \cap K_2) = M$ hold.*

(2) *The factorisations*

$$\Psi_1(K_1) + (\Psi_2(K_2) \cap \Psi_3(K_3)) = \Psi_2(K_2) + (\Psi_1(K_1) \cap \Psi_3(K_3)) = \Psi_3(K_3) + (\Psi_1(K_1) \cap \Psi_2(K_2)) = k\mathbb{Z}_2 \tag{4}$$

hold for some epimorphisms $\Psi_i : \mathsf{N}_M(K_i) \to k\mathbb{Z}_2$ ($i = 1,\ 2,\ 3$) such that Ψ_1, Ψ_2, and Ψ_3 agree on $\mathsf{N}_M(K_1) \cap \mathsf{N}_M(K_2) \cap \mathsf{N}_M(K_3)$.

(3) *The factorisations (4) hold for all epimorphisms $\Psi_i : \mathsf{N}_M(K_i) \to k\mathbb{Z}_2$ ($i = 1,\ 2,\ 3$) such that Ψ_1, Ψ_2, and Ψ_3 agree on $\mathsf{N}_M(K_1) \cap \mathsf{N}_M(K_2) \cap \mathsf{N}_M(K_3)$.*

2. SUBGROUPS OF DIRECT PRODUCTS

If $G_1, \ldots, G_k$ are groups and $G = G_1 \times \cdots \times G_k$, then for $I \subseteq \{1, \ldots, k\}$ the symbol σ_I denotes the projection $G \to \prod_{i \in I} G_i$. If I is a singleton $\{i\}$, then we write $\sigma_I = \sigma_i$, as in the previous section.

Lemma 2.1. *Let G_1 and G_2 be isomorphic groups, $\varphi : G_1 \to G_2$ an isomorphism, K a subgroup of G_1, and H a subgroup of $G_1 \times G_2$. Then*

$$\{(a, \varphi(a)) \mid a \in K\} H = G_1 \times G_2 \tag{5}$$

if and only if

$$\{\varphi^{-1}(b^{-1})a \mid b \in \varphi(K),\ (a, b) \in H\} = G_1. \tag{6}$$

In particular if (5) holds then $(K \cap \varphi^{-1}(\sigma_2(H)))\sigma_1(H) = G_1$. Furthermore in this case, if G_1 is simple and K, $\varphi^{-1}(\sigma_2(H))$, $\sigma_1(H)$ are proper subgroups of G_1, then $\{K, \varphi^{-1}(\sigma_2(H)), \sigma_1(H)\}$ is a strong multiple factorisation of G_1.

Proof. Suppose first that (5) holds. Then for all x in G_1 there exist some $k \in K$, $a \in G_1$, $b \in G_2$ such that $(a, b) \in H$ and $(x, 1) = (k, \varphi(k))(a, b)$. Hence $k = \varphi^{-1}(b^{-1}) \in K \cap \varphi^{-1}(\sigma_2(H))$ and $x = ka = \varphi^{-1}(b^{-1})a$. Thus (6) holds, and also $(K \cap \varphi^{-1}(\sigma_2(H)))\sigma_1(H) = G_1$. This implies that $K\sigma_1(H) = \varphi^{-1}(\sigma_2(H))\sigma_1(H) = G_1$, and, by (5), $K\varphi^{-1}(\sigma_2(H)) = G_1$. If G_1 is simple and the subgroups K, $\varphi^{-1}(\sigma_2(H))$, and $\sigma_1(H)$ are proper, then Lemma 4.3 of Baddeley and Praeger [2] yields that $\{K, \varphi^{-1}(\sigma_2(H)), \sigma_1(H)\}$ is a strong multiple factorisation of G_1.

Conversely, if (6) holds then for all $g \in G_1$ there is some $(a,b) \in H$ such that $b \in \varphi(K)$ and $\varphi^{-1}(b^{-1})a = g$. Then $(g,1) = (\varphi^{-1}(b^{-1}), b^{-1})(a,b)$, and so the first coordinate subgroup of $G_1 \times G_2$ is contained in the left hand side of (5). If $g_2 \in G_2$ then there exists some $g_1 \in G_1$ such that $\varphi(g_1) = g_2^{-1}$. By (6) there is some $(a,b) \in H$ such that $b \in \varphi(K)$ and $\varphi^{-1}(b^{-1})a = g_1$, and so $b^{-1}\varphi(a) = g_2^{-1}$, which yields $\varphi(a^{-1})b = g_2$. Then $(a^{-1}, \varphi(a^{-1}))(a,b) = (1, g_2)$, so the second coordinate subgroup is also contained in the left hand side of (5), and so (5) holds. □

The following useful lemma is well-known and is due to Goursat; see Theorem 1.6.1 in Schmidt [9].

Lemma 2.2. *Suppose that G_1, G_2 are groups. Let N_1 and N_2 be normal subgroups of G_1 and G_2, respectively, such that $G_1/N_1 \cong G_2/N_2$, and let $\varphi : G_1/N_1 \to G_2/N_2$ be an isomorphism. Then the set*

$$H(\varphi) = \{(g_1, g_2) \mid g_1 \in G_1,\ g_2 \in G_2,\ \varphi(g_1N_1) = g_2N_2\} \tag{7}$$

is a subdirect subgroup of $G_1 \times G_2$. Moreover $H : \varphi \mapsto H(\varphi)$ is a bijection between the set of isomorphisms between quotients of G_1 and G_2 and the set of subdirect subgroups of $G_1 \times G_2$.

The following lemma is useful for proving Theorems 1.2, 1.5, and 1.7.

Lemma 2.3. *Let $G_1, \ldots, G_k$ be finite groups, and suppose that for $i = 1, \ldots, k$, N_i is a perfect subgroup of G_i. Set $G = G_1 \times \cdots \times G_k$ and let K be a subgroup of G such that for all i_1, i_2 with $1 \leqslant i_1 < i_2 \leqslant k$, we have $N_{i_1} \times N_{i_2} \leqslant \sigma_{\{i_1,i_2\}}(K)$. Then $N_1 \times \cdots \times N_k \leqslant K$.*

Proof. We prove by induction on m that for all $i_1, \ldots, i_m$ such that $1 \leqslant m \leqslant k$ and $1 \leqslant i_1 < \cdots < i_m \leqslant k$, we have $N_{i_1} \times \cdots \times N_{i_m} \leqslant \sigma_{\{i_1,\ldots,i_m\}}(K)$. Note that for $m = k$, this yields the required result. By assumption this condition holds for $m = 2$. Suppose that $k \geqslant 3$, and the condition holds for $m-1 \leqslant k-1$, and let us prove it for m. Without loss of generality we show that $N_1 \times \cdots \times N_m \leqslant \sigma_{\{1,\ldots,m\}}(K)$. Let a and b be elements of N_1. Then, since $m \geqslant 3$, the induction hypothesis applies to the projections $\sigma_{\{1,3,\ldots,m\}}$ and $\sigma_{\{1,2,4,\ldots,m\}}$, and we obtain

$$(a, 1, \ldots, 1) \in \sigma_{\{1,3,\ldots,m\}}(K) \quad \text{and} \quad (b, 1, \ldots, 1) \in \sigma_{\{1,2,4,\ldots,m\}}(K).$$

Hence there are elements $c \in G_2$ and $d \in G_3$ such that

$$(a, c, 1, \ldots, 1),\ (b, 1, d, 1, \ldots, 1) \in \sigma_{\{1,\ldots,m\}}(K),$$

and so their commutator $([a, b], 1, \ldots, 1)$ is also an element of $\sigma_{\{1,\ldots,m\}}(K)$. Therefore $\sigma_{\{1,\ldots,m\}}(K)$ contains all commutators $[a, b]$ where a, $b \in N_1$. This amounts to saying that $N_1' = N_1 \leqslant \sigma_{\{1,\ldots,m\}}(K)$. Similar argument shows that $N_i \leqslant \sigma_{\{1,\ldots,m\}}(K)$ for $i = 2, \ldots, m$. The lemma now follows by induction. □

3. LS-factorisations

The major results of this paper are concerned with factorisations of finite groups where each of the factors has a quotient of prime order. To assist with our proofs we introduce the following terminology.

Definition 3.1. *Let G be a finite group. Then a set $\{(L_1, S_1), (L_2, S_2)\}$ is called an* LS-factorisation *of G, if L_1, L_2, S_1, S_2 are subgroups of G, and there is some prime p such that $S_i \lhd L_i$, $|L_i : S_i| = p$ for $i \in \{1, 2\}$, and $L_1L_2 = L_1S_2 = S_1L_2 = G$, but $S_1S_2 \neq G$. The number p is called the* index *of the LS-factorisation.*

For example, let G be a group isomorphic to $\mathsf{Sp}_4(q)$ with some even $q \geqslant 4$, and let A and B be non-conjugate subgroups both isomorphic to $\mathsf{Sp}_2(q^2)\cdot 2$. Then $\{(A, A'), (B, B')\}$ is an LS-factorisation of G with index 2; see row 5 of Table 1.

Also, if G is an almost simple group with socle T of prime index such that $G = L_1L_2 = (L_1 \cap T)L_2 = L_1(L_2 \cap T)$ for some subgroups L_1 and L_2 not containing and not contained in T, then $\{(L_1, L_1 \cap T), (L_2, L_2 \cap T)\}$ is an LS-factorisation of G. There are many examples of this kind; see [6]. Later in the paper we will use the two examples and a non-example given by the following lemma.

Lemma 3.2. *Let $T \cong \mathsf{Sp}_6(2)$ and A, B, and C be subgroups of T such that $\{A, B, C\}$ is a strong multiple factorisation of T, and $A \cong \mathsf{G}_2(2)$, $B \cong \mathsf{O}_6^-(2)$, $C \cong \mathsf{O}_6^+(2)$. Then*

$$A' \cap B \cap C = A \cap B' \cap C = A \cap B \cap C' = A' \cap B' \cap C',$$

and $|A \cap B \cap C : A' \cap B' \cap C'| = 2$. Moreover, $T = A'B'$, and both $\{(A, A'), (C, C')\}$ and $\{(B, B'), (C, C')\}$ are LS-factorisations of T with index 2.

Proof. Most statements of this lemma are proved by Baddeley and Praeger [2] on pages 183–184. Here we only have to show that $A'B' = T$. This follows once we show that $A'\cap B\neq A\cap B'$. For if this holds then both $A'\cap B$ and $A\cap B'$ are normal subgroups of $A\cap B$ with index 2, and $(A'\cap B)(A\cap B')=A\cap B$. Hence

$$|A'\cap B:A'\cap B'|=|A'\cap B:(A'\cap B)\cap(A\cap B')|=|(A'\cap B)(A\cap B'):A\cap B'|=2.$$

Thus $|A\cap B:A'\cap B'|=4$ and

$$|A'B'|=\frac{|A'|\cdot|B'|}{|A'\cap B'|}=\frac{|A|\cdot|B|}{|A\cap B|}=|T|.$$

Therefore $A'B'=T$.

So let us prove that $A'\cap B\neq A\cap B'$. As $AB=T$, the index of $A\cap B$ in A is 28. From page 14 of the Atlas [4] we obtain $A\cap B\cong X\rtimes\mathbb{Z}_8\rtimes\mathbb{Z}_2$, and that $A'\cap B\cong X\rtimes\mathbb{Z}_8$ where X is an extraspecial group of order 27. In particular $A'\cap B$ contains an element of order 8. Now the index of $A\cap B$ in B is 120, and, using the information on page 26 of the Atlas, we obtain that $A\cap B$ must be contained in a maximal subgroup of B which is isomorphic to $X\rtimes 2\mathsf{S}_4$, and $A\cap B'$ must be contained in a maximal subgroup of B' which is isomorphic to $X\rtimes 2\mathsf{A}_4$. Since the Sylow 2-subgroup of A_4 is elementary abelian, $A\cap B'$ has no element with order 8. Thus $A'\cap B\neq A\cap B'$. □

Lemma 3.3. *Let G be a finite group and $\{(L_1,S_1),(L_2,S_2)\}$ an LS-factorisation of G of index p. Then $L_1\cap S_2=S_1\cap L_2=S_1\cap S_2\lhd L_1\cap L_2$, $|L_1\cap L_2:S_1\cap S_2|=p$, and $|S_1S_2|=|G|/p$. Also $L_i=(L_1\cap L_2)S_i$ for $i=1,\ 2$.*

Proof. From the given factorisations we obtain

$$\frac{|L_1|\cdot|L_2|}{|L_1\cap L_2|}=\frac{|L_1|\cdot|S_2|}{|L_1\cap S_2|}=\frac{|S_1|\cdot|L_2|}{|S_1\cap L_2|}=|G|. \tag{8}$$

Then $|L_1\cap L_2:L_1\cap S_2|=|L_1\cap L_2:S_1\cap L_2|=p$. As $G=L_2S_1$, we obtain

$$\frac{|G|}{|S_1S_2|}=\frac{|L_2S_1S_2|}{|S_1S_2|}=\frac{|L_2|}{|(S_1S_2)\cap L_2|}=\frac{|L_2|}{|(S_1\cap L_2)S_2|}.$$

Recall that S_2 is a normal subgroup of L_2, and so $S_1\cap L_2$ normalises S_2. Thus $(S_1\cap L_2)S_2$ is a subgroup of L_2 containing S_2. As $|L_2:S_2|=p$, we have $|L_2:(S_1\cap L_2)S_2|\in\{1,p\}$. Then $S_1S_2\neq G$ implies that $|L_2:(S_1\cap L_2)S_2|=p$ and $|S_1S_2|=|G|/p$. Therefore

$$\frac{|G|}{p}=|S_1S_2|=\frac{|S_1|\cdot|S_2|}{|S_1\cap S_2|}. \tag{9}$$

As $|L_1:S_1|=|L_2:S_2|=p$ the combination of (9) and (8) yields that

$$S_1\cap S_2=L_1\cap S_2=S_1\cap L_2$$

is a normal subgroup of $L_1 \cap L_2$ with index p.

For $i = 1,\ 2$, the equation $(L_1 \cap L_2)S_i = L_i$ follows from Dedekind's modular law. $\square$

The following result contains the essence of Theorem 1.4.

Proposition 3.4. *Suppose that for* $i = 1, \dots, k$, G_i *is a finite group with an LS-factorisation* $\left\{\left(L_1^{(i)}, S_1^{(i)}\right), \left(L_2^{(i)}, S_2^{(i)}\right)\right\}$ *of index* p, *and set* $G = G_1 \times \cdots \times G_k$. *For* $i = 1,\ 2$, *let* $L_i = L_i^{(1)} \times \cdots \times L_i^{(k)}$, $S_i = S_i^{(1)} \times \cdots \times S_i^{(k)}$, *let* K_i *be a subgroup of* G *such that* $S_i \leqslant K_i \leqslant L_i$, *and let* Ψ_i *be an epimorphism* $L_i \to k\mathbb{Z}_p$ *with kernel* S_i *such that* Ψ_1 *and* Ψ_2 *agree on* $L_1 \cap L_2$. *Then* $K_1K_2 = G$ *if and only if*

$$\Psi_1(K_1) + \Psi_2(K_2) = k\mathbb{Z}_p. \tag{10}$$

Proof. As $S_1 \leqslant K_1$ and $S_2 \leqslant K_2$, we have $|K_1| = |S_1|\cdot|\Psi_1(K_1)|$ and $|K_2| = |S_2|\cdot|\Psi_2(K_2)|$. We claim that

$$\Psi_1(K_1 \cap K_2) = \Psi_1(K_1) \cap \Psi_1(L_1 \cap K_2).$$

Since $K_1 \cap K_2 = K_1 \cap (L_1 \cap K_2)$, we certainly have $\Psi_1(K_1 \cap K_2) \leqslant \Psi_1(K_1) \cap \Psi_1(L_1 \cap K_2)$. Let us show that that $\Psi_1(K_1) \cap \Psi_1(L_1 \cap K_2) \leqslant \Psi_1(K_1 \cap K_2)$. Choose an element $x \in \Psi_1(K_1) \cap \Psi_1(L_1 \cap K_2)$. Then there are elements $a \in K_1$ and $b \in L_1 \cap K_2$ such that $x = \Psi_1(a) = \Psi_1(b)$. Hence $ab^{-1} \in \ker \Psi_1 = S_1 \leqslant K_1$, and so $b \in K_1 \cap K_2$, that is $x = \Psi_1(b) \in \Psi_1(K_1 \cap K_2)$. Therefore

$$\Psi_1(K_1 \cap K_2) = \Psi_1(K_1) \cap \Psi_1(L_1 \cap K_2) = \Psi_1(K_1) \cap \Psi_2(L_1 \cap K_2).$$

Now $K_2 = K_2 \cap (L_1S_2) = (L_1 \cap K_2)S_2$ and $\Psi_2(S_2) = 0$, and so $\Psi_2(K_2) = \Psi_2(L_1 \cap K_2)$. Thus

$$\Psi_1(K_1 \cap K_2) = \Psi_1(K_1) \cap \Psi_2(K_2).$$

Note that $S_1 \cap S_2 = \ker \Psi_1|_{K_1 \cap K_2}$, and so $|K_1 \cap K_2| = |S_1 \cap S_2| \cdot |\Psi_1(K_1) \cap \Psi_2(K_2)|$. Hence

$$\begin{aligned}|K_1K_2| = \frac{|K_1| \cdot |K_2|}{|K_1 \cap K_2|} &= \frac{|S_1| \cdot |\Psi_1(K_1)| \cdot |S_2| \cdot |\Psi_2(K_2)|}{|S_1 \cap S_2| \cdot |\Psi_1(K_1) \cap \Psi_2(K_2)|} \\ &= |\Psi_1(K_1) + \Psi_2(K_2)| \cdot |S_1S_2| = \frac{|\Psi_1(K_1) + \Psi_2(K_2)| \cdot |G|}{p^k},\end{aligned}$$

by Lemma 3.3. Now if (10) holds, then $|K_1K_2| = |k\mathbb{Z}_p| \cdot |G|/p^k = |G|$, so $G = K_1K_2$. Conversely if $G = K_1K_2$, then

$$|\Psi_1(K_1) + \Psi_2(K_2)| = \frac{p^k \cdot |K_1K_2|}{|G|} = p^k,$$

and hence (10) holds. $\square$

4. Full factorisations

In this section we prove Theorems 1.2 and 1.4. We start with a small technical lemma concerning some of the groups in Tables 1 and 2.

Lemma 4.1. (a) *Let* $T \cong \mathsf{P}\Omega_8^+(2)$ *and* A *be a subgroup of* T *isomorphic to* $\mathbb{Z}_2^6 \rtimes \mathsf{A}_7$ *or* $\mathbb{Z}_2^6 \rtimes \mathsf{A}_8$*. Then* A *is perfect and has a unique minimal normal subgroup isomorphic to* $\mathbb{Z}_2^6$*.*

(b) *Let* $T \cong \mathsf{P}\Omega_8^+(3)$ *and let* B *be a subgroup of* T *isomorphic to* $\mathbb{Z}_3^6 \rtimes \mathsf{PSL}_4(3)$*. Then* B *is perfect and has a unique minimal normal subgroup isomorphic to* $\mathbb{Z}_3^6$*.*

Proof. (a) Suppose first that $A \cong \mathbb{Z}_2^6 \rtimes \mathsf{A}_7$. From the character tables for T in the Atlas ([4], page 85) we obtain that T has a unique conjugacy class of elements of order 7, and the centraliser of such an element $x \in A$ is the cyclic subgroup $\langle x \rangle$. Hence x acts fixed-point freely on $\mathbb{Z}_2^6$. Since A_7 is simple it follows that A_7 acts faithfully on $\mathbb{Z}_2^6$, and in particular $\mathbb{Z}_2^6$ is self centralising in A. If V is a non-trivial proper $\langle x \rangle$-submodule of $\mathbb{Z}_2^6$, then, as x is fix-point free, we have that 7 divides $|V|-1$. Hence $\dim V = 3$. If V were A-invariant, then, as A_7 is simple, we would have that A_7 is isomorphic to a subgroup of $\mathsf{GL}_3(2)$, which is not the case, so no such submodule exists. Hence $\mathbb{Z}_2^6$ is a minimal normal subgroup of A, and it is also a minimal normal subgroup of A when $A \cong \mathbb{Z}_2^6 \rtimes \mathsf{A}_8$. As in both cases $\mathbb{Z}_2^6$ is self-centralising, it is the unique minimal normal subgroup of A. Thus $\mathbb{Z}_2^6 \leqslant A'$, and $\mathsf{A}_7 \leqslant A'$, and so $A = A'$.

(b) Using the Atlas ([4], page 140), we find that T has two conjugacy classes of elements with order 13, and the centraliser of such an element x in both classes is the cyclic subgroup $\langle x \rangle$. The proof of part (b) is now analogous to that given for part (a). □

Proof of Theorem 1.2. Recall that $(T_i, \{\sigma_i(K_1), \sigma_i(K_2)\})$ is a full factorisation, and hence the subgroups $\sigma_i(K_1)$, $\sigma_i(K_2)$ are listed in Table 1. Using Lemma 4.1, we can read off from this table that $\sigma_i(K_j)'$ is a perfect group for all i and j. In the light of Lemma 2.3 we only have to show that for all i_1, i_2, and j such that $1 \leqslant i_1 < i_2 \leqslant k$ and $j \in \{1,2\}$,

$$\sigma_{i_1}(K_j)' \times \sigma_{i_2}(K_j)' \leqslant \sigma_{\{i_1,i_2\}}(K_j). \tag{11}$$

We argue by contradiction and assume that K_1 does not satisfy (11) for some i_1 and i_2. Then we may suppose without loss of generality that $M = T_1 \times T_2 = T^2$ where T is a finite simple group, K_1 and K_2 are such that $(M, \{K_1, K_2\})$ is a full factorisation, $\sigma_1(K_1)' \times \sigma_2(K_1)' \not\leqslant K_1$, and $K_2 = \sigma_1(K_2) \times \sigma_2(K_2)$ such that $\sigma_1(K_2)$ and $\sigma_2(K_2)$ are maximal subgroups of T_1 and T_2, respectively. Inspection of Table 1 and Lemma 4.1 show that for $i \in \{1,2\}$ one of the following holds for $\sigma_i(K_1)$:

1. $\sigma_i(K_1)$ is simple;
2. $\sigma_i(K_1)$ is almost simple such that $\sigma_i(K_1)/\operatorname{Soc}\sigma_i(K_1) \cong \mathbb{Z}_2$;
3. $\sigma_i(K_1)$ is perfect and it has a unique minimal normal subgroup, which is elementary abelian, such that $\sigma_i(K_1)/\operatorname{Soc}\sigma_i(K_1)$ is non-abelian and simple.

Lemma 2.2 implies that in all cases $K_1 = H(\varphi)$ where φ is an isomorphism between two quotients of $\sigma_1(K_1)$ and $\sigma_2(K_1)$. Combining Lemma 2.2 with the fact that (11) fails, we obtain that one of the following holds.

(a) Both $\sigma_1(K_1)$ and $\sigma_2(K_1)$ are almost simple groups, and $K_1 = H(\varphi)$ where φ is an isomorphism $\sigma_1(K_1) \to \sigma_2(K_1)$, so $K_1 \cong \sigma_1(K_1) \cong \sigma_2(K_1)$.

(b) Both $\sigma_1(K_1)$ and $\sigma_2(K_1)$ are affine groups, and, by Lemma 4.1, $K_1 = H(\varphi)$ where φ is an isomorphism mapping either $\sigma_1(K_1) \to \sigma_2(K_1)$ or $\sigma_1(K_1)/\operatorname{Soc}\sigma_1(K_1) \to \sigma_2(K_1)/\operatorname{Soc}\sigma_2(K_1)$.

For a prime p and finite group G, let $|G|_p$ denote the exponent of the largest p-power dividing $|G|$. As $K_1K_2 = T_1 \times T_2$, we have $|K_1| \cdot |K_2|/|K_1 \cap K_2| = |T|^2$, and hence $|T|^2$ is a divisor of $|K_1| \cdot |K_2|$. Therefore for any prime p we have

$$2|T|_p \leqslant |K_1|_p + |K_2|_p. \tag{12}$$

In the rest of the proof we obtain contradictions for each of the possible groups T.

$\boxed{T \cong \mathsf{A}_6}$ Suppose that $T \cong \mathsf{A}_6$ and $\sigma_i(K_j) \cong \mathsf{A}_5$ for all $i,\ j \in \{1,2\}$. Then by the previous discussion we have $K_1 \cong \mathsf{A}_5$ and $K_2 \cong \mathsf{A}_5 \times \mathsf{A}_5$. However $|K_1|_3 + |K_2|_3 = 3$, while $2|T|_3 = 4$, which contradicts (12).

$\boxed{T \cong \mathsf{M}_{12}}$ Here $\sigma_i(K_j)$ is simple and is isomorphic to either M_{11} or $\mathsf{PSL}_2(11)$. Then K_1 is also isomorphic to either M_{11} or to $\mathsf{PSL}_2(11)$, and K_2 is isomorphic to one of $\mathsf{M}_{11}\times\mathsf{M}_{11}$, $\mathsf{M}_{11}\times\mathsf{PSL}_2(11)$, or $\mathsf{PSL}_2(11)\times\mathsf{PSL}_2(11)$. Since $|T|_3 = 3$, $|\mathsf{M}_{11}|_3 = 2$, and $|\mathsf{PSL}_2(11)|_3 = 1$, it follows that $K_1 \cong \mathsf{M}_{11}$, and $K_2 \cong \mathsf{M}_{11} \times \mathsf{M}_{11}$. Thus there are subgroups $A,\ C \leqslant T_1$ and $B,\ D \leqslant T_2$, and an isomorphism $\alpha : A \to B$ such that $K_1 = \{(a, \alpha(a)) \mid a \in A\}$ and $K_2 = C \times D$. Since the subgroups of M_{12} isomorphic to M_{11} form a single orbit under the action of $\mathsf{Aut}(\mathsf{M}_{12})$ and $\mathsf{Out}(\mathsf{M}_{11}) = 1$, we have that α can be extended to an isomorphism $\bar\alpha : T_1 \to T_2$. By Lemma 2.1, this implies that $\{A, C, \bar\alpha^{-1}(D)\}$ is a strong multiple factorisation of T_1. Using the list of strong multiple factorisations [2], we find that such a strong multiple factorisation does not exist, which is a contradiction.

$\boxed{T = \mathsf{P}\Omega_8^+(q),\ q \geqslant 3}$ Here $\sigma_i(K_j) \cong \Omega_7(q)$ for all $i,\ j \in \{1,2\}$. By the results of Kleidman [5], the subgroups of $\mathsf{P}\Omega_8^+(q)$ isomorphic to $\Omega_7(q)$ form a single orbit under $\mathsf{Aut}(\mathsf{P}\Omega_8^+(q))$, and all automorphisms of $\Omega_7(q)$ are induced by automorphisms of $\mathsf{P}\Omega_8^+(q)$. Hence the argument for the case $T \cong \mathsf{M}_{12}$, $K_1 \cong \mathsf{M}_{11}$, and $K_2 \cong \mathsf{M}_{11} \times \mathsf{M}_{11}$ above yields a contradiction.

$\boxed{T \cong \mathsf{P}\Omega_8^+(2)}$ Here $2|T|_5 = 4$, while $|\sigma_i(K_j)|_5 = 1$ for all i and j. Hence $|K_2|_5 = 2$ and for all possibilities of K_1, $|K_1|_5 = 1$, contradicting (12).

$\boxed{T \cong \mathsf{Sp}_4(q),\ q \geqslant 4,\ q \text{ even}}$ Here $\sigma_i(K_j)$ is almost simple, and hence $K_1 \cong \mathsf{Sp}_2(q^2) \cdot 2$ or $K_1 \cong \mathsf{Sp}_2(q^2)$ and $K_2 \cong (\mathsf{Sp}_2(q^2) \cdot 2) \times (\mathsf{Sp}_2(q^2) \cdot 2)$. Then

$$|T| = q^4(q^4-1)(q^2-1), \quad |K_1| \mid 2 \cdot q^2(q^4-1), \quad \text{and} \quad |K_2| = 4 \cdot q^4(q^4-1)^2.$$

If $q = 2^\ell$ then $2|T|_2 = 8\ell$, while $|K_1|_2 + |K_2|_2 \leqslant 6\ell + 3$. Since $\ell \geqslant 2$, this contradicts inequality (12). □

Proof of Corollary 1.3. In lines 1–3 the corollary is an immediate consequence of Theorem 1.2 and the fact that both A and B are non-abelian simple, and hence perfect, groups. The same is true for line 4 unless $B \cong \mathsf{S}_7$ or S_8. In this case, by the results of Baddeley and Praeger [2], we have $A'B' = T$ and the corollary follows. □

Now we turn to the proof of Theorem 1.4. For the rest of the section we make the following assumptions. Set $M = T_1 \times \cdots \times T_k$ where the T_i are isomorphic to $\mathsf{Sp}_4(q)$ with some even $q \geqslant 4$. Let K_1 and K_2 be subgroups of M such that $(T_i, \{\sigma_i(K_1), \sigma_i(K_2)\})$ is a full factorisation and (1) holds, and set $L_i = \mathsf{N}_M(K_i)$ and $S_i = L_i'$ for $i = 1,\ 2$. Then for all i,

$$S_i = \sigma_1(K_i)' \times \cdots \times \sigma_k(K_i)' \leqslant K_i \leqslant \mathsf{N}_{T_1}(\sigma_1(K_i)) \times \cdots \times \mathsf{N}_{T_k}(\sigma_k(K_i)) = L_i.$$

Lemma 4.2. *If Ψ_1 is an epimorphism $L_1 \to k\mathbb{Z}_2$, then there exists a unique epimorphism $\Psi_2 : L_2 \to k\mathbb{Z}_2$ such that Ψ_1 and Ψ_2 agree on $L_1 \cap L_2$.*

Proof. As L_1/S_1 is the unique quotient of L_1 isomorphic to $k\mathbb{Z}_2$, $\ker \Psi_1 = S_1$. If $x \in L_2$ then, as $L_2 = (L_1 \cap L_2)S_2$ by Lemma 3.3, we have $x = x'x''$ where $x' \in L_1 \cap L_2$ and $x'' \in S_2$. Define $\Psi_2 : L_2 \to k\mathbb{Z}_2$ by $\Psi_2(x) = \Psi_1(x')$. We claim that Ψ_2 is well-defined. If $x = x'x'' = y'y''$ such that $x',\ y' \in L_1 \cap L_2$ and $x'',\ y'' \in S_2$, then $(y')^{-1}x' \in L_1 \cap S_2 = S_1 \cap S_2$. Hence $(y')^{-1}x' \in S_1$, and so $\Psi_1(x') = \Psi_1(y')$. Thus

Ψ_2 is well-defined, and it is straightforward to check that it is a homomorphism. As $L_1=(L_1\cap L_2)S_1$ and Ψ_1 is an epimorphism, we have $\Psi_2(L_1\cap L_2)=\Psi_1(L_1\cap L_2)=k\mathbb{Z}_2$, and Ψ_2 is also an epimorphism. The unique quotient of L_2 isomorphic to $k\mathbb{Z}_2$ is L_2/S_2, and hence $\ker\Psi_2=S_2$. Now let $x\in L_1\cap L_2$. Then, by definition, $\Psi_2(x)=\Psi_1(x)$. Thus Ψ_2 agrees with Ψ_1 on $L_1\cap L_2$.

Let us now show that if Ψ_2' is another epimorphism that has the required properties, then $\Psi_2'=\Psi_2$. Let $x\in L_2$. Then, as above, we have $x=x'x''$ for some $x'\in L_1\cap L_2$ and $x''\in S_2$. As $\Psi_2'(x'')=0$, we obtain $\Psi_2'(x)=\Psi_2'(x')=\Psi_1(x')=\Psi_2(x)$. □

Proof of Theorem 1.4. By our assumptions, $\{(\sigma_i(L_1),\sigma_i(S_1)),(\sigma_i(L_2),\sigma_i(S_2))\}$ is an LS-factorisation of T_i for all i. If $K_1K_2=M$, then Proposition 3.4 implies that $\Psi_1(K_1)+\Psi_2(K_2)=k\mathbb{Z}_2$ for all $\Psi_1:L_1\to k\mathbb{Z}_2$ and $\Psi_2:L_2\to k\mathbb{Z}_2$ epimorphisms which agree on $L_1\cap L_2$. Hence Statement 3 holds. Lemma 4.2 implies that if Statement 3 holds, then so does Statement 2. If Statement 2 holds for some Ψ_1 and Ψ_2, then Proposition 3.4 implies that $K_1K_2=M$. □

Lemma 4.2 shows that the maps Ψ_1 and Ψ_2 can be constructed in many different ways, but we find the maps in the following example natural.

Example 4.3. *For $i=1,\ 2$, define $\Psi_i:L_i\to k\mathbb{Z}_2$ by $\Psi_i(x)=(y_1,\ldots,y_k)$ where $y_j=1$ if and only if $\sigma_j(x)\in\sigma_j(L_i)\setminus\sigma_j(S_i)$. Then we claim that $\Psi_1|_{L_1\cap L_2}=\Psi_2|_{L_1\cap L_2}$. For let $x=(x_1,\ldots,x_k)\in L_1\cap L_2$, and suppose that $\Psi_1(x)=(y_1,\ldots,y_k)$ and $\Psi_2(x)=(z_1,\ldots,z_k)$. If $y_i=0$ for some i, then $x_i\in\sigma_i(S_1)$. By Lemma 3.3 $\sigma_i(S_1)\cap\sigma_i(L_2)=\sigma_i(S_1)\cap\sigma_i(S_2)$, and so $x_i\in\sigma_i(S_1)\cap\sigma_i(S_2)$. Thus $z_i=0$. Similarly, $z_i=0$ implies that $y_i=0$.*

5. STRONG MULTIPLE FACTORISATIONS

In this section we prove Theorems 1.7 and 1.9 and Corollary 1.8.

Proof of Theorem 1.7. Suppose that $(M,\{K_1,K_2,K_3\})$ is a strong multiple factorisation. Note that for all i and j, the group $\sigma_i(K_j)'$ is perfect (see Table 2 and Lemma 4.1). Hence using Lemma 2.3, it suffices to prove that for all $i_1,\ i_2$ such that $1\leqslant i_1<i_2\leqslant k$, we have

$$\sigma_{i_1}(K_i)'\times\sigma_{i_2}(K_i)'\leqslant\sigma_{\{i_1,i_2\}}(K_i)\quad\text{for}\quad i=1,\ 2,\ 3.$$

Suppose to the contrary that this is not the case. We may assume without loss of generality that $M=T_1\times T_2=T^2$ where T is a finite simple group, and that $\sigma_1(K_1)'\times\sigma_2(K_1)'\not\leqslant K_1$,

$K_2 = \sigma_1(K_2) \times \sigma_2(K_2)$, and $K_3 = \sigma_1(K_3) \times \sigma_2(K_3)$ where $\sigma_1(K_2)$, $\sigma_1(K_3)$, and $\sigma_2(K_2)$, $\sigma_2(K_3)$ are all maximal subgroups of T_1 and T_2, respectively. Inspection of Table 2 and Lemma 4.1 show that for $i \in \{1,2\}$ one of the following holds for $\sigma_i(K_1)$:

1. $\sigma_i(K_1)$ is simple;
2. $\sigma_i(K_1)$ is almost simple such that $\sigma_i(K_1)/\operatorname{Soc}\sigma_i(K_1) \cong \mathbb{Z}_2$;
3. $\sigma_i(K_1)$ is perfect, and it has a unique minimal normal subgroup, which is elementary abelian, such that $\sigma_i(K_1)/\operatorname{Soc}\sigma_i(K_1)$ is non-abelian and simple.

Lemma 2.2 implies that in all cases $K_1 = H(\varphi)$ where φ is an isomorphism between two quotients of $\sigma_1(K_1)$ and $\sigma_2(K_1)$. Using Lemma 2.2, we obtain that one of the following holds.

(a) Both $\sigma_1(K_1)$ and $\sigma_2(K_1)$ are almost simple groups, and $K_1 = H(\varphi)$ where φ is an isomorphism $\sigma_1(K_1) \to \sigma_2(K_1)$.

(b) Both $\sigma_1(K_1)$ and $\sigma_2(K_1)$ are affine groups, and $K_1 = H(\varphi)$ where φ is an isomorphism either between $\sigma_1(K_1)$ and $\sigma_2(K_1)$, or between $\sigma_1(K_1)/\operatorname{Soc}\sigma_1(K_1)$ and $\sigma_2(K_1)/\operatorname{Soc}\sigma_2(K_1)$.

As $K_1(K_2 \cap K_3) = T_1 \times T_2$, we have $|K_1| \cdot |K_2 \cap K_3|/|K_1 \cap K_2 \cap K_3| = |T|^2$. Since $K_2K_3 = T_1 \times T_2$, we have $|K_2 \cap K_3| = |K_2| \cdot |K_3|/|T|^2$, and so $|K_1| \cdot |K_2| \cdot |K_3|/|K_1 \cap K_2 \cap K_3| = |T|^4$. Thus $|K_1|_p + |K_2|_p + |K_3|_p \geqslant 4|T|_p$ for all primes p where $|\cdot|_p$ is defined in the proof of Theorem 1.2. In the rest of the proof we show case by case that this scenario is not possible.

$\boxed{T \cong \mathsf{Sp}_6(2)}$ Here $4|T|_2 = 4 \cdot 9 = 36$, while $|\sigma_i(K_j)|_2 \leqslant 7$ for all i and j. Thus, as $\sigma_i(K_j)$ is almost simple, $|K_1|_2 = |\sigma_i(K_1)|_2 \leqslant 7$ and $|K_2|_2$, $|K_3|_2 \leqslant 14$, which is a contradiction.

$\boxed{T \cong \mathsf{Sp}_{4a}(2),\ a \geqslant 2}$ Here $\sigma_i(K_j)$ is isomorphic to $\mathsf{Sp}_{2a}(4)\cdot 2$, $\mathsf{O}^+_{4a}(2)$, or $\mathsf{O}^-_{4a}(2)$, and

$$\begin{aligned}
|T| = |\mathsf{Sp}_{4a}(2)| &= 2^{4a^2}\left(2^{4a}-1\right)\left(2^{4a-2}-1\right)\cdots\left(2^2-1\right),\\
|\mathsf{Sp}_{2a}(4)\cdot 2| &= 2\cdot 4^{a^2}\left(4^{2a}-1\right)\left(4^{2a-2}-1\right)\cdots\left(4^2-1\right),\\
|\mathsf{O}^+_{4a}(2)| &= 2^{4a^2-2a+1}\left(2^{2a}-1\right)\left(2^{4a-2}-1\right)\left(2^{4a-4}-1\right)\cdots\left(2^2-1\right),\\
|\mathsf{O}^-_{4a}(2)| &= 2^{4a^2-2a+1}\left(2^{2a}+1\right)\left(2^{4a-2}-1\right)\left(2^{4a-4}-1\right)\cdots\left(2^2-1\right).
\end{aligned}$$

Since all groups involved as $\sigma_i(K_j)$ are almost simple, we have $|K_1|_p = |\sigma_1(K_1)|_p$ for all primes p. By Zsigmondy's Theorem [10] (see also [7] and 2.4 of [6]), $2^{4a}-1$ has a prime divisor, p say, which does not divide 2^j-1 for $1 \leqslant j \leqslant 4a-1$. If $b = |T|_p$, then $4|T|_p = 4b$.

If $K_1 \cong \mathsf{Sp}_{2a}(4)\cdot 2$ or $K_1 \cong \mathsf{O}^-_{4a}(2)$ then $|K_1|_p = b$ and $|K_2|_p + |K_3|_p = 2b$, which is a contradiction. Hence $K_1 \cong \mathsf{O}^+_{4a}(2)$ and $|K_1|_2 = 4a^2-2a+1$ and $|K_2|_2+|K_3|_2 = 12a^2-4a+4$. As $4|T|_2 = 16a^2$, this is also a contradiction.

$\boxed{T \cong \mathsf{P}\Omega^+_8(3)}$ Here $4|T|_5 = 8$, $|A_1|_5 = 1$, $|A_2|_5 = 1$, and $|A_3|_5 = 2$, where A_1, A_2 and A_3 are as in Table 2. If $K_1 = H(\varphi)$ where $\varphi : \sigma_1(K_1) \to \sigma_2(K_1)$, then $|K_1|_5 = |\sigma_1(K_1)|_5$. If $K_1 = H(\varphi)$ where $\varphi : \sigma_1(K_1)/\operatorname{Soc}\sigma_1(K_1) \to \sigma_2(K_1)/\operatorname{Soc}\sigma_2(K_1)$, then $\sigma_1(K_1) \cong \sigma_2(K_1) \cong \mathbb{Z}_3^6 \rtimes \mathsf{PSL}_4(3)$ and $|\operatorname{Soc}\sigma_1(K_1)|_5 = |\operatorname{Soc}\sigma_2(K_1)|_5 = 0$, so again $|K_1|_5 = |\sigma_1(K_1)|_5$. So $|K_1|_5 = |\sigma_1(K_1)|_5$ always holds, and $|K_1|_5 \leqslant 2$. If $|K_1|_5 = 2$ then $|K_2|_5 = |K_3|_5 = 2$, and so $|K_1|_5+|K_2|_5+|K_3|_5 = 6$, which is a contradiction. If $|K_1|_5 = 1$ then $|K_2|_5 + |K_3|_5 = 6$, and so $|K_1|_5 + |K_2|_5 + |K_3|_5 = 7$, which is also a contradiction. □

Proof of Corollary 1.8. If $T \cong \mathsf{P}\Omega^+_8(3)$ then the statement of the corollary is an immediate consequence of Theorem 1.7, because the projections $\sigma_i(K_1)$, $\sigma_i(K_2)$, $\sigma_i(K_3)$ are perfect groups for all i. Thus we may assume that $M = T^k = T_1 \times \cdots \times T_k$ where $T \cong \mathsf{Sp}_{4a}(2)$, and $\{K_1, K_2, K_3\}$ is a strong multiple factorisation of M. It is sufficient to prove that $K_1 = \sigma_1(K_1) \times \cdots \times \sigma_k(K_1)$. Assume without loss of generality that K_2 and K_3 are direct products of their projections, and these projections are maximal subgroups of the T_i. Let $\bar{K}_1 = \mathsf{N}_M(K_1)$. By Theorem 1.7, $\sigma_1(K_1)' \times \cdots \times \sigma_k(K_1)' \leqslant K_1$, and so $\bar{K}_1 = \mathsf{N}_{T_1}(\sigma_1(K_1)) \times \cdots \times \mathsf{N}_{T_k}(\sigma_k(K_1))$. In particular $\bar{K}_1$ is a direct product of its projections, and they are maximal subgroups of the T_i. Then $\bar{K}_1(K_2 \cap K_3) = M$, and hence

$$\frac{|\bar{K}_1|\cdot|K_2\cap K_3|}{|\bar{K}_1\cap K_2\cap K_3|} = |M|.$$

Note that

$$\bar{K}_1 \cap K_2 \cap K_3 = (\sigma_1(K_1)\cap\sigma_1(K_2)\cap\sigma_1(K_3)) \times \cdots \times (\sigma_k(K_1)\cap\sigma_k(K_2)\cap\sigma_k(K_3)),$$

and it is proved by Baddeley and Praeger [2] on pages 181–182 that

$$\sigma_i(K_1)\cap\sigma_i(K_2)\cap\sigma_i(K_3) = \sigma_i(K_1)'\cap\sigma_i(K_2)'\cap\sigma_i(K_3)'$$

for all i, and therefore

$$\begin{aligned}(\sigma_1(K_1)\cap\sigma_1(K_2)\cap\sigma_1(K_3)) \times \cdots \times (\sigma_k(K_1)\cap\sigma_k(K_2)\cap\sigma_k(K_3))\\ = (\sigma_1(K_1)'\cap\sigma_1(K_2)'\cap\sigma_1(K_3)') \times \cdots \times (\sigma_k(K_1)'\cap\sigma_k(K_2)'\cap\sigma_k(K_3)'),\end{aligned}$$

which forces $K_1 \cap K_2 \cap K_3 = \bar{K}_1 \cap K_2 \cap K_3$. Since $K_1(K_2 \cap K_3) = M$, we have

$$\frac{|K_1|\cdot|K_2\cap K_3|}{|K_1\cap K_2\cap K_3|} = |M|,$$

which implies that $|K_1| = |\bar{K}_1|$, and hence $K_1 = \bar{K}_1$. □

Now we turn to the proof of Theorem 1.9. In the remainder of this section we make the following assumptions: $M = T_1 \times \cdots \times T_k \cong T^k$ where $T \cong \mathsf{Sp}_6(2)$; $K_1,\ K_2,\ K_3 \leqslant M$ such that $(T_i, \{\sigma_i(K_1), \sigma_i(K_2), \sigma_i(K_3)\})$ is a strong multiple factorisation for $i = 1, \ldots, k$; and K_j contains $\prod_{i=1}^{k} \sigma_i(K_j)'$ for $j = 1,\ 2,\ 3$. Set $S_j = K_j'$ and $L_j = \mathsf{N}_M(K_j)$. It follows from Table 2 that $S_j = L_j' = K_j'$, and L_j/S_j is the unique quotient of L_j isomorphic to $k\mathbb{Z}_2$.

First we show that the epimorphisms Ψ_2 and Ψ_3 in Theorem 1.9 are uniquely determined by Ψ_1.

Lemma 5.1. *If Ψ_1 is an epimorphism $L_1 \to k\mathbb{Z}_2$, then there are unique epimorphisms $\Psi_2 : L_2 \to k\mathbb{Z}_2$ and $\Psi_3 : L_3 \to k\mathbb{Z}_2$ such that Ψ_1, Ψ_2, and Ψ_3 agree on $L_1 \cap L_2 \cap L_3$.*

Proof. As L_1/S_1 is the unique quotient of L_1 isomorphic to $k\mathbb{Z}_2$, we have $\ker \Psi_1 = S_1$. Let $x \in L_2$ and using the equation $L_2 = (L_1 \cap L_2 \cap L_3)S_2$, write $x = x'x''$ for some $x' \in L_1 \cap L_2 \cap L_3$ and $x'' \in S_2$. Then define $\Psi_2(x) = \Psi_1(x')$. First we show that Ψ_2 is well-defined. Let $x = x'x'' = y'y''$ with $x',\ y' \in L_1 \cap L_2 \cap L_3$ and $x'',\ y'' \in S_2$. Then $(y')^{-1}x' = y''(x'')^{-1} \in L_1 \cap S_2 \cap L_3 = S_1 \cap S_2 \cap S_3 \leqslant S_1$, and so $\Psi_1(x') = \Psi_1(y')$. Thus Ψ_2 is well-defined, and it is not hard to check that it is a homomorphism. Since $L_1 = (L_1 \cap L_2 \cap L_3)S_1$, we have $\Psi_2(L_2) \geqslant \Psi_2(L_1 \cap L_2 \cap L_3) = \Psi_1(L_1 \cap L_2 \cap L_3) = \Psi_1(L_1) = k\mathbb{Z}_2$, and Ψ_2 is an epimorphism. An epimorphism Ψ_3 can be defined in a similar manner, and $\ker \Psi_i = S_i$ for $i = 1,\ 2$. Finally if $x \in L_1 \cap L_2 \cap L_3$, then by definition $\Psi_1(x) = \Psi_2(x) = \Psi_3(x)$. Thus the maps Ψ_2 and Ψ_3 have the required properties.

Let $\Psi_2' : L_2 \to k\mathbb{Z}_2$ and $\Psi_3' : L_3 \to k\mathbb{Z}_2$ also have these properties. Let $x \in L_2$ and write $x = x'x''$ for some $x' \in L_1 \cap L_2 \cap L_3$ and $x'' \in S_2$. Then, since Ψ_2' has kernel S_2 and agrees with Ψ_1 on $L_1 \cap L_2 \cap L_3$, $\Psi_2'(x) = \Psi_2'(x') = \Psi_1(x') = \Psi_2(x)$. Thus $\Psi_2 = \Psi_2'$. The equality $\Psi_3 = \Psi_3'$ can be shown similarly. □

Next we show that the abelian factorisations in equation (4) of Theorem 1.9 are sufficient for the factorisations $M = K_iK_j$ to hold for different i and j.

Lemma 5.2. *If Statement 2 of Theorem 1.9 holds, then $K_iK_j = M$ for all different i and j.*

Proof. Suppose that Statement 2 in Theorem 1.9 holds for some maps Ψ_1, Ψ_2, Ψ_3. Without loss of generality, it suffices to show that $K_1K_2 = M$. Let I be the subset of $\{1,\ldots,k\}$ such that $i \in I$ if and only if $\mathsf{N}_T(\sigma_i(K_3)) \cong \mathsf{O}_6^+(2)$. Then, as $\sigma_i(K_1)'\sigma_i(K_2)' = T_i$ for all $i \in I$ (see Lemma 3.2), we obtain $\prod_{i\in I} T_i \leqslant K_1K_2$. Now let $J = \{1,\ldots,k\}\setminus I$, and let σ_J denote the projection map $\sigma_J : M \to \prod_{i\in J} T_i$. Then it suffices to prove that $\sigma_J(K_1)\sigma_J(K_2) = \sigma_J(M)$. Without loss of generality assume that $J = \{1,\ldots,k'\}$ for some $k' \leqslant k$. For a subgroup $X \leqslant M$ such that $X = \prod_{i=1}^k \sigma_i(X)$ and a subset $\mathcal{I} \subseteq \{1,\ldots,k\}$ we write $X^{\mathcal{I}} = \prod_{i\in\mathcal{I}} \sigma_i(X)$. Using this notation, we have $L_i = L_i^J \times L_i^I$ and $S_i = S_i^J \times S_i^I$ for $i = 1,\ 2,\ 3$. For $i = 1,\ 2,\ 3$, let the maps $\Psi_i^J : L_i^J \to k\mathbb{Z}_2$ and $\Psi_i^I : L_i^I \to k\mathbb{Z}_2$ be defined as the restriction of Ψ_i to L_i^J and L_i^I, respectively. We prove that the maps Ψ_1^J and Ψ_2^J satisfy the conditions of Proposition 3.4. The same proposition will then imply that $\sigma_J(K_1)\sigma_J(K_2) = \sigma_J(M)$.

For all i,

$$k\mathbb{Z}_2 = \Psi_i(L_i) = \Psi_i\left(L_i^J L_i^I\right) = \Psi_i^J\left(L_i^J\right) + \Psi_i^I\left(L_i^I\right).$$

Now $\ker\Psi_i^J = S_i^J$ and $\ker\Psi_i^I = S_i^I$. Moreover $|L_i^J : S_i^J| = 2^{k'}$ and $|L_i^I : S_i^I| = 2^{k-k'}$, and so we must have $k\mathbb{Z}_2 = \Psi_i^J\left(L_i^J\right) \oplus \Psi_i^I\left(L_i^I\right)$ and $\Psi_i^J(L_i^J) \cong k'\mathbb{Z}_2$.

Recall that the epimorphisms Ψ_1 and Ψ_2 agree on $L_1 \cap L_2 \cap L_3$. However, as $\sigma_j(S_1)\cap\sigma_j(S_2) = \sigma_j(S_1)\cap\sigma_j(L_2)$ for all $j \in J$ (see Lemmas 3.2 and 3.3), we obtain

$$L_3^J(S_1^J \cap S_2^J) = L_3^J(S_1^J \cap L_2^J) = M^J,$$

which yields $L_1^J \cap L_2^J = (L_1^J \cap L_2^J \cap L_3^J)(S_1^J \cap S_2^J)$. Note that $S_1^J \cap S_2^J = \ker\Psi_1^J \cap \ker\Psi_2^J$, and so Ψ_1^J and Ψ_2^J must agree on $L_1^J \cap L_2^J = (L_1\cap L_2)^J$.

Now $L_i = (L_1 \cap L_2)S_i$ for all $i \in \{1,2\}$. Therefore for the same i we have $L_i^J = (L_1\cap L_2)^J S_i^J$. As $\ker\Psi_i^J = S_i^J$, we have $\Psi_i^J\left(L_i^J\right) = \Psi_i^J\left((L_1\cap L_2)^J\right)$. As $\Psi_1^J|_{(L_1\cap L_2)^J} = \Psi_2^J|_{(L_1\cap L_2)^J}$, this implies that $\Psi_1^J(L_1^J) = \Psi_2^J(L_2^J)$.

Thus Ψ_1^J and Ψ_2^J are epimorphisms onto the same group $\Psi_1^J(L_1^J) = \Psi_2^J(L_2^J) \cong k'\mathbb{Z}_2$ such that $\Psi_1^J|_{L_1^J\cap L_2^J} = \Psi_2^J|_{L_1^J\cap L_2^J}$. Moreover, $\{(\sigma_j(L_1),\sigma_j(S_1)),(\sigma_j(L_2),\sigma_j(S_2))\}$ is an LS-factorisation of T_j for all $j \in J$. Hence, by Proposition 3.4, $\sigma_J(K_1)\sigma_J(K_2) = M^J$ provided that

$$\Psi_1^J(\sigma_J(K_1)) + \Psi_2^J(\sigma_J(K_2)) = \Psi_1^J(L_1^J) \cong k'\mathbb{Z}_2. \tag{13}$$

It remains to prove therefore that (13) holds. Now $k\mathbb{Z}_2 = \Psi_1(K_1) + \Psi_2(K_2)$ and, as $K_i \leqslant \sigma_J(K_i) \times L_i^I$, it follows that

$$k\mathbb{Z}_2 = \Psi_1^J(\sigma_J(K_1)) + \Psi_2^J(\sigma_J(K_2)) + \Psi_1^I(L_1^I) + \Psi_2^I(L_2^I). \tag{14}$$

We claim that $\Psi_1^I(L_1^I) = \Psi_2^I(L_2^I)$. For let $x \in \Psi_1^I(L_1^I)$; then there is some $y \in L_1^I$ such that $x = \Psi_1^I(y)$. As $L_1^I = (L_1^I \cap L_2^I \cap L_3^I)S_1^I$, we have $y = y'y''$ for some $y' \in L_1^I \cap L_2^I \cap L_3^I$ and $y'' \in S_1^I$, and, as $y'' \in \ker \Psi_1^I$,

$$\Psi_1^I(y) = \Psi_1^I(y') \in \Psi_1^I(L_1^I \cap L_2^I \cap L_3^I) = \Psi_2^I(L_1^I \cap L_2^I \cap L_3^I) \leqslant \Psi_2^I(L_2^I).$$

Thus $\Psi_1^I(L_1^I) \leqslant \Psi_2^I(L_2^I)$. Similar argument shows that $\Psi_2^I(L_2^I) \leqslant \Psi_1^I(L_1^I)$, and so $\Psi_1^I(L_1^I) = \Psi_2^I(L_2^I)$. Hence (14) can be written as

$$k\mathbb{Z}_2 = \Psi_1^J(\sigma_J(K_1)) + \Psi_2^J(\sigma_J(K_2)) + \Psi_1^I(L_1^I).$$

Since $k\mathbb{Z}_2 = \Psi_i^J(L_i^J) \oplus \Psi_i^I(L_i^I)$ for $i = 1,\ 2$, and $\Psi_1^J(L_1^J) = \Psi_2^J(L_2^J)$, it follows that

$$k'\mathbb{Z}_2 = \Psi_i^J(L_i^J) = \Psi_1^J(\sigma_J(K_1)) + \Psi_2^J(\sigma_J(K_2)),$$

as required. □

Now we are ready to prove Theorem 1.9.

Proof of Theorem 1.9. By Lemma 5.1, Statement 3 implies Statement 2. In the following we show that Statement 2 implies Statement 1, and Statement 1 implies Statement 3.

Let $\Psi_i : L_i \to k\mathbb{Z}_2$ be an epimorphism for $i = 1,\ 2,\ 3$, such that the Ψ_i agree on $L_1 \cap L_2 \cap L_3$. Then, as already noted, $\ker \Psi_i = S_i$ for all i. Since $S_i \leqslant K_i$, $|K_i| = |S_i| \cdot |\Psi_i(K_i)|$ for all i. Before assuming that Statement 1, 2 or 3 holds, we obtain an expression for the order of $K_1 \cap K_2 \cap K_3$ involving the Ψ_i. First we claim that

$$\Psi_1(K_1 \cap K_2 \cap K_3) = \Psi_1(K_1) \cap \Psi_1(L_1 \cap K_2 \cap K_3).$$

Indeed, $\Psi_1(K_1 \cap K_2 \cap K_3) = \Psi_1(K_1 \cap (L_1 \cap K_2 \cap K_3)) \leqslant \Psi_1(K_1) \cap \Psi_1(L_1 \cap K_2 \cap K_3)$. So we only have to prove that $\Psi_1(K_1) \cap \Psi_1(L_1 \cap K_2 \cap K_3) \leqslant \Psi_1(K_1 \cap K_2 \cap K_3)$. Let $x \in \Psi_1(K_1) \cap \Psi_1(L_1 \cap K_2 \cap K_3)$. Then $x = \Psi_1(a) = \Psi_1(b)$ for some $a \in K_1$ and $b \in L_1 \cap K_2 \cap K_3$. Then $ab^{-1} \in \ker \Psi_1 = S_1 \leqslant K_1$, and so $b \in K_1 \cap K_2 \cap K_3$. Thus $x = \Psi_1(b) \in \Psi_1(K_1 \cap K_2 \cap K_3)$. Similar argument yields that

$$\Psi_2(L_1 \cap K_2 \cap K_3) = \Psi_2(K_2) \cap \Psi_2(L_1 \cap L_2 \cap K_3).$$

Thus our assumptions on the Ψ_i imply that

$$\begin{aligned}\Psi_1(K_1\cap K_2\cap K_3) &= \Psi_1(K_1)\cap\Psi_1(L_1\cap K_2\cap K_3) = \Psi_1(K_1)\cap\Psi_2(L_1\cap K_2\cap K_3)\\ &= \Psi_1(K_1)\cap\Psi_2(K_2)\cap\Psi_2(L_1\cap L_2\cap K_3) = \Psi_1(K_1)\cap\Psi_2(K_2)\cap\Psi_3(L_1\cap L_2\cap K_3).\end{aligned}$$

Clearly, $\Psi_3(L_1\cap L_2\cap K_3)\leqslant\Psi_3(K_3)$. On the other hand, as $(L_1\cap L_2)S_3=M$, we have $K_3=(L_1\cap L_2\cap K_3)S_3$. Thus $\Psi_3(K_3)\leqslant\Psi_3(L_1\cap L_2\cap K_3)$, and consequently $\Psi_3(K_3)=\Psi_3(L_1\cap L_2\cap K_3)$. Hence

$$\Psi_1(K_1\cap K_2\cap K_3)=\Psi_1(K_1)\cap\Psi_2(K_2)\cap\Psi_3(K_3).$$

Note that $S_1\cap S_2\cap S_3=\ker\Psi_1|_{K_1\cap K_2\cap K_3}$, and so

$$|K_1\cap K_2\cap K_3|=|S_1\cap S_2\cap S_3|\cdot|\Psi_1\left(K_1\right)\cap\Psi_2\left(K_2\right)\cap\Psi_3\left(K_3\right)|.$$

Now let i, j, and m be indices such that $\{i,j,m\}=\{1,2,3\}$. Then

$$\begin{aligned}|K_i(K_j\cap K_m)| &= \frac{|K_i|\cdot|K_j\cap K_m|}{|K_i\cap K_j\cap K_m|}=\frac{|K_i|\cdot|K_j|\cdot|K_m|}{|K_jK_m|\cdot|K_i\cap K_j\cap K_m|}\\ &= \frac{|S_i|\cdot|\Psi_i\left(K_i\right)|\cdot|S_j|\cdot|\Psi_j\left(K_j\right)|\cdot|S_m|\cdot|\Psi_m\left(K_m\right)|}{|K_jK_m|\cdot|S_i\cap S_j\cap S_m|\cdot|\Psi_i\left(K_i\right)\cap\Psi_j\left(K_j\right)\cap\Psi_m\left(K_m\right)|}.\end{aligned}$$

Also by Lemma 3.2,

$$\begin{aligned}\frac{|S_i|\cdot|S_j|\cdot|S_m|}{|S_i\cap S_j\cap S_m|} &= \frac{|L_i|}{2^k}\cdot\frac{|L_j|}{2^k}\cdot\frac{|L_m|}{2^k}\cdot\frac{2^k}{|L_i\cap L_j\cap L_m|}\\ &= \frac{|L_i|\cdot|L_j|}{|L_i\cap L_j|}\cdot\frac{|L_i\cap L_j|\cdot|L_m|}{|L_i\cap L_j\cap L_m|}\cdot\frac{1}{2^{2k}}=\frac{|M|^2}{2^{2k}},\end{aligned}$$

and hence

$$|K_i(K_j\cap K_m)|=\frac{|M|^2}{2^{2k}\cdot|K_jK_m|}\cdot\frac{|\Psi_i(K_i)|\cdot|\Psi_j(K_j)|\cdot|\Psi_m(K_m)|}{|\Psi_i(K_i)\cap\Psi_j(K_j)\cap\Psi_m(K_m)|}. \tag{15}$$

If Statement 2 of Theorem 1.9 holds for the Ψ_i, then

$$2^k=\frac{|\Psi_i\left(K_i\right)|\cdot|\Psi_j\left(K_j\right)\cap\Psi_m\left(K_m\right)|}{|\Psi_i\left(K_i\right)\cap\Psi_j\left(K_j\right)\cap\Psi_m\left(K_m\right)|}=\frac{|\Psi_i\left(K_i\right)|\cdot|\Psi_j\left(K_j\right)|\cdot|\Psi_m\left(K_m\right)|}{2^k\cdot|\Psi_i\left(K_i\right)\cap\Psi_j\left(K_j\right)\cap\Psi_m\left(K_m\right)|}$$

and, by Lemma 5.2, $K_jK_m=M$. Then (15) implies that

$$|K_i(K_j\cap K_m)|=2^{2k}\frac{|M|^2}{2^{2k}\cdot|M|}=|M|.$$

Thus $K_i(K_j\cap K_m)=M$. As i, j, and m were chosen arbitrarily, Statement 2 implies that Statement 1 holds.

Conversely, suppose that Statement 1 holds. Then using (15), we get

$$\frac{|\Psi_i(K_i)|\cdot|\Psi_j(K_j)|}{|\Psi_i(K_i)\cap\Psi_j(K_j)|}\cdot\frac{|\Psi_m(K_m)|\cdot|\Psi_i(K_i)\cap\Psi_j(K_j)|}{|\Psi_i(K_i)\cap\Psi_j(K_j)\cap\Psi_m(K_m)|}$$
$$=\frac{|\Psi_i(K_i)|\cdot|\Psi_j(K_j)|\cdot|\Psi_m(K_m)|}{|\Psi_i(K_i)\cap\Psi_j(K_j)\cap\Psi_m(K_m)|}=\frac{|K_i(K_j\cap K_m)|\cdot 2^{2k}\cdot|K_jK_m|}{|M|^2}=2^{2k}. \tag{16}$$

Note that

$$\frac{|\Psi_i(K_i)|\cdot|\Psi_j(K_j)|}{|\Psi_i(K_i)\cap\Psi_j(K_j)|}\leqslant 2^k \quad\text{and}\quad \frac{|\Psi_m(K_m)|\cdot|\Psi_i(K_i)\cap\Psi_j(K_j)|}{|\Psi_i(K_i)\cap\Psi_j(K_j)\cap\Psi_m(K_m)|}\leqslant 2^k.$$

Hence equality in (16) is only possible when

$$\frac{|\Psi_i(K_i)\cap\Psi_j(K_j)|\cdot|\Psi_m(K_m)|}{|\Psi_i(K_i)\cap\Psi_j(K_j)\cap\Psi_m(K_m)|}=2^k,$$

that is $(\Psi_i(K_i)\cap\Psi_j(K_j))+\Psi_m(K_m)=k\mathbb{Z}_2$. Thus Statement 1 implies Statement 3. □

Note that Lemma 5.1 implies that one can construct many sets $\{\Psi_1,\Psi_2,\Psi_3\}$ of maps with the required properties, but a set of natural ones can be constructed as in Example 4.3.

6. The proof of Theorem 1.5

In this section we prove Theorem 1.5. Before we start we introduce a piece of notation. Let $G_1,\dots,G_k,H_1,\dots,H_k$ be groups, and for $i=1,\dots,k$ let $\psi_i: G_i\to H_i$ be a homomorphism. Then we define the homomorphism $(\psi_1\times\cdots\times\psi_k): G_1\times\cdots\times G_k\to H_1\times\cdots\times H_k$ as

$$(\psi_1\times\cdots\times\psi_k)(g_1,\dots,g_k)=(\psi_1(g_1),\dots,\psi_k(g_k))$$

for all $g_1\in G_1,\dots,g_k\in G_k$.

Proof of Theorem 1.5. (a) Suppose that $DK=M$. First we prove that the ordered pair $(T_i,\{\sigma_i(K),\varphi_i^{-1}(\sigma_{i+k}(K))\})$ is a full factorisation for all $i\in\{1,\dots,k\}$. By the assumptions of the theorem, we have $\sigma_{\{i,i+k\}}(D)\sigma_{\{i,i+k\}}(K)=T_i\times T_{i+k}$. Then by Lemma 2.1, $\{\sigma_i(K),\varphi_i^{-1}(\sigma_{i+k}(K))\}$ is a factorisation of T_i. As the subgroups of this factorisation are isomorphic, we have that it is a full factorisation.

Let us now prove that $\sigma_1(K)'\times\cdots\times\sigma_{2k}(K)'\leqslant K$ holds. Inspection of Table 1 shows that $\sigma_i(K)'$ is perfect for all i, and so, by Lemma 2.3, it suffices to show that for all i_1, i_2

such that $1 \leqslant i_1 < i_2 \leqslant 2k$,

$$\sigma_{i_1}(K)' \times \sigma_{i_2}(K)' \leqslant \sigma_{\{i_1,i_2\}}(K). \tag{17}$$

Suppose that this is not the case, and choose i_1, i_2 such that (17) does not hold. Then, as $\sigma_{i_1}(K)$ and $\sigma_{i_2}(K)$ are almost simple, we have $\sigma_{\{i_1,i_2\}}(K) \cong \sigma_{i_1}(K) \cong \sigma_{i_2}(K)$. Note that if $\sigma_{\{i_1,i_2\}}(D) \cong T_1$, then $|\sigma_{\{i_1,i_2\}}(D)| \cdot |\sigma_{\{i_1,i_2\}}(K)| < |T_1|^2$, and so $\sigma_{\{i_1,i_2\}}(D)\sigma_{\{i_1,i_2\}}(K) \neq T_{i_1} \times T_{i_2}$. Thus we may assume without loss of generality that $\sigma_{\{i_1,i_2\}}(D) = T_{i_1} \times T_{i_2}$, and $i_2 \neq i_1 + k$. According to part (a) and Lemma 2.1,

$$(T_1 \times \cdots \times T_k, \{\sigma_{\{1,\ldots,k\}}(K), (\varphi_1 \times \cdots \times \varphi_k)^{-1}(\sigma_{\{k+1,\ldots,2k\}}(K))\})$$

is a full factorisation, and hence, by Theorem 1.2,

$$\sigma_1(K)' \times \cdots \times \sigma_k(K)' \leqslant \sigma_{\{1,\ldots,k\}}(K) \quad \text{and} \quad \sigma_{k+1}(K)' \times \cdots \times \sigma_{2k}(K)' \leqslant \sigma_{\{k+1,\ldots,2k\}}(K).$$

Thus $i_1 \in \{1, \ldots, k\}$, $i_2 \in \{k+1, \ldots, 2k\}$, and $i_2 \neq i_1 + k$. Now

$$\begin{aligned}\sigma_{\{i_1,i_2,i_1+k,i_2-k\}}(D) &= \{(a_1, a_2, \varphi_{i_1}(a_1), \varphi_{i_2}^{-1}(a_2)) \mid a_1 \in T_{i_1},\ a_2 \in T_{i_2}\} \\ &= \{((a_1, a_2), (\varphi_{i_1} \times \varphi_{i_2}^{-1})(a_1, a_2)) \mid a_1 \in T_{i_1},\ a_2 \in T_{i_2}\},\end{aligned}$$

and

$$\sigma_{\{i_1,i_2,i_1+k,i_2-k\}}(D)\sigma_{\{i_1,i_2,i_1+k,i_2-k\}}(K) = T_{i_1} \times T_{i_2} \times T_{i_1+k} \times T_{i_2-k}.$$

Hence, by Lemma 2.1, $\sigma_{\{i_1,i_2\}}(K)((\varphi_{i_1}^{-1} \times \varphi_{i_2})(\sigma_{\{i_1+k,i_2-k\}}(K))) = T_{i_1} \times T_{i_2}$. Since the projections of $\sigma_{\{i_1,i_2\}}(K)$ and $(\varphi_{i_1}^{-1} \times \varphi_{i_2})(\sigma_{\{i_1+k,i_2-k\}}(K))$ are isomorphic,

$$(T_{i_1} \times T_{i_2}, \{\sigma_{\{i_1,i_2\}}(K), (\varphi_{i_1}^{-1} \times \varphi_{i_2})(\sigma_{\{i_1+k,i_2-k\}}(K))\})$$

is a full factorisation, but $\sigma_{\{i_1,i_2\}}(K)$ does not satisfy (1) in Theorem 1.2. This is a contradiction, and hence $\sigma_1(K)' \times \cdots \times \sigma_{2k}(K)' \leqslant K$.

(b) This statement easily follows from the fact that if T, A and B are as in one of the rows 1–4 in Table 1 such that $A \cong B$, then A and B are simple groups.

(c) Suppose that $DK = M$. Set $L = \mathsf{N}_M(K)$ and $S = L'$. Note that $L \cong (\mathsf{Sp}_2(q^2){\cdot}2)^{2k}$ and $S \cong \mathsf{Sp}_2(q^2)^{2k}$. Then $L = (DK) \cap L = (L \cap D)K$. Hence for all epimorphisms $\Psi : L \to 2k\mathbb{Z}_2$ we have

$$2k\mathbb{Z}_2 = \Psi(L) = \Psi((L \cap D)K) = \Psi(L \cap D) + \Psi(K).$$

Thus Statement 1 implies Statement 3.

Suppose now that $\Psi(L\cap D)+\Psi(K) = 2k\mathbb{Z}_2$ holds for some epimorphism $\Psi : L \to 2k\mathbb{Z}_2$. Then $\ker\Psi = S$, and hence $|K| = |S|\cdot|\Psi(K)|$. We claim that

$$\Psi(K\cap D) = \Psi(K)\cap\Psi(L\cap D).$$

Indeed, $\Psi(K\cap D) = \Psi(K\cap L\cap D) \leqslant \Psi(K)\cap\Psi(L\cap D)$. Let $x\in\Psi(K)\cap\Psi(L\cap D)$. Then there is some $a\in K$ and $d\in L\cap D$ such that $x = \Psi(a) = \Psi(d)$, and so $ad^{-1}\in S\leqslant K$. Since $a\in K$, this implies that $d\in K$, and hence $d\in K\cap D$. That is $x = \Psi(d)\in\Psi(K\cap D)$.

Therefore

$$|\Psi(K\cap D)| = |\Psi(K)\cap\Psi(L\cap D)|,$$

and hence $|K\cap D| = |S\cap D|\cdot|\Psi(K)\cap\Psi(L\cap D)|$. Thus

$$\frac{|K|\cdot|D|}{|K\cap D|} = \frac{|S|\cdot|\Psi(K)|\cdot|D|}{|S\cap D|\cdot|\Psi(K)\cap\Psi(L\cap D)|}.$$

As $\Psi(K)+\Psi(L\cap D) = 2k\mathbb{Z}_2$, we have

$$\frac{|\Psi(K)|\cdot|\Psi(L\cap D)|}{|\Psi(K)\cap\Psi(L\cap D)|} = 2^{2k}.$$

As $\ker\Psi|_{L\cap D} = S\cap D$, we obtain $\Psi(L\cap D) = 2^k$, and so

$$\frac{|\Psi(K)|}{|\Psi(K)\cap\Psi(L\cap D)|} = 2^k.$$

Then

$$\frac{|K|\cdot|D|}{|K\cap D|} = \frac{2^k\cdot|S|\cdot|D|}{|S\cap D|}.$$

Let

$$\bar{S} = \sigma_1(L)\times\cdots\times\sigma_k(L)\times\sigma_{k+1}(S)\times\cdots\times\sigma_{2k}(S).$$

Then $|\bar{S}| = 2^k\cdot|S|$, and

$$\begin{aligned}\bar{S}\cap D &= \{(a_1,\dots,a_k,\varphi_1(a_1),\dots,\varphi_k(a_k)) \mid a_i\in\sigma_i(\bar{S})\cap\varphi_i^{-1}(\sigma_{i+k}(\bar{S}))\}\\ &= \{(a_1,\dots,a_k,\varphi_1(a_1),\dots,\varphi_k(a_k)) \mid a_i\in\sigma_i(S)\cap\varphi_i^{-1}(\sigma_{i+k}(S))\} = S\cap D.\end{aligned}$$

Moreover for all $i\in\{1,\dots,k\}$, Lemma 2.1 implies that $\sigma_{\{i,i+k\}}(D)\sigma_{\{i,i+k\}}(\bar{S}) = T_i\times T_{i+k}$, and so $\bar{S}D = M$. Hence

$$\frac{|K|\cdot|D|}{|K\cap D|} = \frac{2^k\cdot|S|\cdot|D|}{|S\cap D|} = \frac{|\bar{S}|\cdot|D|}{|\bar{S}\cap D|} = |\bar{S}D| = |M|,$$

and therefore $KD = M$. Thus Statement 2 implies Statement 1.

Finally, the fact that Statement 3 implies Statement 2 follows from the fact that $L/S \cong 2k\mathbb{Z}_2$. □

References

[1] Robert W. Baddeley and Cheryl E. Praeger, 'On primitive overgroups of quasiprimitive permutation groups', Technical report 1996/14, Department of Mathematics and Computer Science, University of Leicester, 1996.

[2] Robert W. Baddeley and Cheryl E. Praeger, 'On classifying all full factorisations and multiple-factorisations of the finite almost simple groups', *J. Algebra* 204 (1998) 129–187.

[3] Robert W. Baddeley, Cheryl E. Praeger, and Csaba Schneider, 'Permutation groups preserving Cartesian decompositions', *In progress.*

[4] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *Atlas of finite groups* (Oxford University Press, Eynsham, 1985).

[5] Peter B. Kleidman, 'The maximal subgroups of the finite 8-dimensional orthogonal groups $P\Omega_8^+(q)$ and of their automorphism groups', *J. Algebra* 110 (1987) 173–242.

[6] Martin W. Liebeck, Cheryl E. Praeger, and Jan Saxl, 'The maximal factorizations of the finite simple groups and their automorphism groups', *Mem. Amer. Math. Soc.* 86 (1990).

[7] Heinz Lüneburg 'Ein einfacher Beweis für den Satz von Zsigmondy über primitive Primteiler von $A^N - 1$', *Geometries and groups (Berlin, 1981)*, Lecture Notes in Mathematics 893 (Springer, Berlin, 1981) 219–222.

[8] Cheryl E. Praeger, 'The inclusion problem for finite primitive permutation groups', *Proc. London Math. Soc.* (3) 60 (1990) 68–88.

[9] Roland Schmidt, *Subgroup lattices of groups* (Walter de Gruyter & Co., Berlin, 1994).

[10] K. Zsigmondy, 'Zür Theorie der Potenzreste', *Monatsh. Math. Phys.* 3 (1892) 265–284.

Department of Mathematics and Statistics, The University of Western Australia, 35 Stirling Highway 6009 Crawley, Western Australia

E-mail address: `praeger@maths.uwa.edu.au`; *WWW:* `www.maths.uwa.edu.au/~praeger`

E-mail address: `csaba@maths.uwa.edu.au`; *WWW:* `www.maths.uwa.edu.au/~csaba`