PERMUTATION GROUPS AND CARTESIAN DECOMPOSITIONS

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Cartesian decompositions of sets

Ω is a finite set.

Cartesian decomposition of Ω:

$$\{(\Omega_i,\alpha_i)\}_{i=1}^{\ell}=\{(\Omega_1,\alpha_1),\ldots,(\Omega_\ell,\alpha_\ell)\}$$

where Ω_i sets and $\alpha_i:\Omega\to\Omega_i$, such that

$$\omega\mapsto(\alpha_1(\omega),\ldots,\alpha_\ell(\omega))$$

is a bijection $\Omega\to\Omega_1\times\cdots\times\Omega_\ell$.

$\{(\Omega_i,\alpha_i)\}_{i=1}^{\ell}$ is equivalent to $\{(\Omega_i',\alpha_i')\}_{i=1}^{\ell}$ if there is $\pi\in\mathsf{S}_\ell$ and $\beta_i:\Omega_i\to\Omega_{i\pi}'$ bijections such that

$$\alpha_{i\pi}'(\omega)=\beta_i(\alpha_i(\omega))\quad\text{for all}\quad i,\ \omega.$$

Cartesian decompositions and partitions

Let $\{(\Omega_i, \alpha_i)\}$ be a Cartesian decomposition of Ω. For all i set

$$\Gamma_i = \{\{\omega \in \Omega \mid \alpha_i(\omega) = \xi\} \mid \xi \in \Omega_i\}.$$

Γ_i is a partition of Ω for all i. Moreover

$$|\gamma_1 \cap \cdots \cap \gamma_\ell| = 1 \text{ for all } \gamma_1 \in \Gamma_1, \ldots, \gamma_\ell \in \Gamma_\ell.$$

Define

$$\beta_i : \Omega \to \Gamma_i, \quad \beta_i(\omega) = \gamma_i \text{ where } \omega \in \gamma_i \in \Gamma_i.$$

Then $\{(\Gamma_i, \beta_i)\}$ is a Cartesian decomposition of Ω equivalent to $\{(\Omega_i, \alpha_i)\}$.

Lemma $\{(\Omega_i, \alpha_i)\}$ and $\{(\Delta_i, \delta_i)\}$ are equivalent if and only if they induce the same $\{(\Gamma_i, \beta_i)\}$.

Example

Let $\Omega = \{1, \ldots, 9\}$ and write

1	2	3
4	5	6
7	8	9

The corresponding Cartesian decomposition is

$\Omega_1 = \Omega_2 = \{1, 2, 3\},\ \alpha_1 : \omega \mapsto$ row number of $\omega,\ \alpha_2 : \omega \mapsto$ col number of ω.

The corresponding partitions are

$$\begin{aligned} \Gamma_1 &= \{\{1,2,3\},\{4,5,6\},\{7,8,9\}\} \\ \Gamma_2 &= \{\{1,4,7\},\{2,5,8\},\{3,6,9\}\}. \end{aligned}$$

Cartesian decompositions and group actions

Let $\{(\Omega_i, \alpha_i)\}$ be a Cartesian decomposition of Ω and $g \in \mathrm{Sym}\,\Omega$. Define $\alpha_i^g(\omega) = \alpha_i(\omega^{g^{-1}})$.

Then $\{(\Omega_i, \alpha_i^g)\}$ is a Cartesian decomposition of Ω.

g is said to preserve $\{(\Omega_i, \alpha_i)\}$ if $\{(\Omega_i, \alpha_i^g)\}$ is equivalent to $\{(\Omega_i, \alpha_i)\}$.

If Γ_i and Γ_i' are the partitions corresponding to α_i, α_i^g then $\Gamma_i' = \Gamma_i^g$. Hence g permutes $\{\Gamma_i\}$.

From now on $G \leqslant \mathrm{Sym}\,\Omega$ and $\{\Gamma_i\}$ is a G-invariant set of partitions, such that

$$|\gamma_1 \cap \cdots \cap \gamma_\ell| = 1 \text{ for all } \gamma_1 \in \Gamma_1, \ldots, \gamma_\ell \in \Gamma_\ell.$$

Objectives and Motivation

Objectives:

- Find a theory of invariant Cartesian decompositions;
- Describe invariant Cartesian decompositions in interesting cases.

Motivation:

- Duality with primitive/imprimitive theory;
- Many groups preserve Cartesian decompositions (e.g. product action);
- Quasiprimitive/primitive inclusion problem (X,PA) case.

Cartesian decompositions and subgroups

Let $M \leqslant \mathrm{Sym}\,\Omega$ transitive and $\{\Gamma_i\}$ is an M-invariant Cartesian decomposition of Ω, such that $M_{(\{\Gamma_i\})} = M$.

Fix $\omega \in \Omega$ and set

$$K_i = M_{\gamma_i} \quad \text{where} \quad \omega \in \gamma_i \in \Gamma_i.$$

Then

$$\bigcap_i K_i = M_\omega \quad \text{and} \quad K_i \left(\bigcap_{j \neq i} K_j \right) = M. \qquad (1)$$

Conversely if $\{K_i\}$ is a set of subgroups with (1) then $\{\Gamma_i\}$ is a Cartesian decomposition of Ω where Γ_i is the system of imprimitivity for K_i. Moreover, $M_{(\{\Gamma_i\})} = M$.

$\{K_i\}$ with (1) is called a Cartesian system of subgroups for M.

Embedding Cartesian systems into larger groups

Let $G \leqslant \mathrm{Sym}\,\Omega$ and $M \lhd G$ such that M is transitive and $M_{(\{\Gamma_i\})} = M$. Let $\{K_i\}$ be the corresponding Cartesian system in M.

Theorem $\{\Gamma_i\}$ is G-invariant iff $\{K_i\}$ is G_ω-invariant.

Theorem If M is a non-abelian transitive minimal normal subgroup and $\{\Gamma_i\}$ is G-invariant then $M_{(\{\Gamma_i\})} = M$.

We only consider groups with a non-abelian transitive minimal normal subgroup M (eg. primitive, quasiprimitive, innately transitive groups).

G_ω-invariant Cartesian systems for M

$\Updownarrow$

G-invariant Cartesian decompositions for Ω

New problem: Find factorisations of characteristically simple groups.

Normal Decompositions

Let M be a non-abelian transitive minimal normal subgroup of G.

Suppose that $M = S_1 \times \cdots \times S_\ell$ such that

$$M_\omega = (S_1 \cap M_\omega) \times \cdots \times (S_\ell \cap M_\omega)$$

and $\{S_1, \ldots, S_\ell\}$ is G-invariant. (Warning: The S_i may not be simple.)

Then

$$(M_\omega \cap S_1) \times S_2 \times \cdots \times S_\ell, \qquad \ldots \qquad , S_1 \times \cdots \times S_{\ell-1} \times (M_\omega \cap S_\ell)$$

is a G_ω-invariant Cartesian system for M.

Hence it determines a G-invariant Cartesian decomposition. Such decomposition is said to be normal.

Not all decompositions are normal

Let $G = \mathsf{PGL}_2(9)$ and consider the transitive G-action with degree 36. Then Soc $G \cong \mathsf{PSL}_2(9) \cong \mathsf{A}_6$ is transitive. By previous result (Soc $G)_{(\{\Gamma_i\})} =$ Soc G.

There are $K_1,\ K_2 \leqslant$ Soc G conjugate under G_ω, such that

$$K_1 \cap K_2 = (\text{Soc } G)_\omega \text{ and } K_1 K_2 = \text{Soc } G,$$

therefore

$$[\text{Soc } G : K_1 \cap K_2] = [\text{Soc } G : K_1] \times [\text{Soc } G : K_2]$$

and G preserves this Cartesian decomposition.

Objective: To understand transitive decompositions for groups with a transitive minimal normal subgroup.

Describing other decompositions

Let $G \leqslant \mathrm{Sym}\,\Omega$, and M a non-abelian, transitive, minimal normal subgroup of G.

Then $M = T_1 \times \cdots \times T_k$ for finite simple T_i.

Problem: Find G-invariant (transitive) Cartesian decompositions of Ω.

Look for G_ω-invariant subgroups $K_1, \ldots, K_\ell \leqslant M$ such that

$$\bigcap K_i = M_\omega \quad \text{and} \quad K_i \left(\bigcap_{j \neq i} K_j \right) = M.$$

In general this is is more difficult than finding all factorisations of finite simple groups.

Let $\sigma_i : M \to T_i$ be the projection.

Theorem For all i the projection $\sigma_i(K_j)$ is proper for at most three j (Baddeley & Praeger).

Thus one of the following holds (statements are independent of i):

- $\sigma_i(K_j) \cong T_i$ for all j (DONE – NORMAL);
- $\sigma_i(K_j)$ is proper for exactly 1 j (DONE – NORMAL);
- $\sigma_i(K_j)$ is proper for exactly 2 j (HARD – COMBINATORIAL DESCRIPTION);
- $\sigma_i(K_j)$ is proper for exactly 3 j (DONE);