

# PERMUTATION GROUPS AND CARTESIAN DECOMPOSITIONS

CSABA SCHNEIDER

DEPARTMENT OF MATHEMATICS AND STATISTICS

THE UNIVERSITY OF WESTERN AUSTRALIA

`csaba@maths.uwa.edu.au`

`http://www.maths.uwa.edu.au/~csaba`

Joint work with R. W. Baddeley & C. E. Praeger

## Cartesian decompositions of sets

$\Omega$  is a finite set.

Cartesian decomposition of  $\Omega$ :

$$\{(\Omega_i, \alpha_i)\}_{i=1}^{\ell} = \{(\Omega_1, \alpha_1), \dots, (\Omega_{\ell}, \alpha_{\ell})\}$$

where  $\Omega_i$  sets and  $\alpha_i : \Omega \rightarrow \Omega_i$ , such that

$$\omega \mapsto (\alpha_1(\omega), \dots, \alpha_{\ell}(\omega))$$

is a bijection  $\Omega \rightarrow \Omega_1 \times \dots \times \Omega_{\ell}$ .

$\{(\Omega_i, \alpha_i)\}_{i=1}^{\ell}$  is **equivalent** to  $\{(\Omega'_i, \alpha'_i)\}_{i=1}^{\ell}$  if there is  $\pi \in S_{\ell}$  and  $\beta_i : \Omega_i \rightarrow \Omega'_{i\pi}$  bijections such that

$$\alpha'_{i\pi}(\omega) = \beta_i(\alpha_i(\omega)) \quad \text{for all } i, \omega.$$

## Cartesian decompositions and partitions

Let  $\{(\Omega_i, \alpha_i)\}$  be a Cartesian decomposition of  $\Omega$ . For all  $i$  set

$$\Gamma_i = \{\{\omega \in \Omega \mid \alpha_i(\omega) = \xi\} \mid \xi \in \Omega_i\}.$$

$\Gamma_i$  is a partition of  $\Omega$  for all  $i$ . Moreover

$$|\gamma_1 \cap \cdots \cap \gamma_\ell| = 1 \text{ for all } \gamma_1 \in \Gamma_1, \dots, \gamma_\ell \in \Gamma_\ell.$$

Define

$$\beta_i : \Omega \rightarrow \Gamma_i, \quad \beta_i(\omega) = \gamma_i \text{ where } \omega \in \gamma_i \in \Gamma_i.$$

Then  $\{(\Gamma_i, \beta_i)\}$  is a Cartesian decomposition of  $\Omega$  equivalent to  $\{(\Omega_i, \alpha_i)\}$ .

**Lemma**  $\{(\Omega_i, \alpha_i)\}$  and  $\{(\Delta_i, \delta_i)\}$  are equivalent if and only if they induce the same  $\{(\Gamma_i, \beta_i)\}$ .

## Cartesian decompositions and group actions

Let  $\{(\Omega_i, \alpha_i)\}$  be a Cartesian decomposition of  $\Omega$  and  $g \in \text{Sym } \Omega$ . Define  $\alpha_i^g(\omega) = \alpha_i(\omega^{g^{-1}})$ .

Then  $\{(\Omega_i, \alpha_i^g)\}$  is a Cartesian decomposition of  $\Omega$ .

$g$  is said to preserve  $\{(\Omega_i, \alpha_i)\}$  if  $\{(\Omega_i, \alpha_i^g)\}$  is equivalent to  $\{(\Omega_i, \alpha_i)\}$ .

If  $\Gamma_i$  and  $\Gamma'_i$  are the partitions corresponding to  $\alpha_i, \alpha_i^g$  then  $\Gamma'_i = \Gamma_i^g$ . Hence  $g$  permutes  $\{\Gamma_i\}$ .

From now on  $G \leq \text{Sym } \Omega$  and  $\{\Gamma_i\}$  is a  $G$ -invariant set of partitions, such that

$$|\gamma_1 \cap \cdots \cap \gamma_\ell| = 1 \text{ for all } \gamma_1 \in \Gamma_1, \dots, \gamma_\ell \in \Gamma_\ell.$$

## Objectives and Motivation

### Objectives:

- Find a theory of invariant Cartesian decompositions;
- Describe invariant Cartesian decompositions in interesting cases.

### Motivation:

- Duality with primitive/imprimitive theory;
- Many groups preserve Cartesian decompositions (e.g. product action);
- Quasiprimitive/primitive inclusion problem  $(X, PA)$  case.

## Cartesian decompositions and subgroups

Let  $M \leq \text{Sym } \Omega$  transitive and  $\{\Gamma_i\}$  is an  $M$ -invariant Cartesian decomposition of  $\Omega$ , such that  $M_{(\{\Gamma_i\})} = M$ .

Fix  $\omega \in \Omega$  and set

$$K_i = M_{\gamma_i} \quad \text{where } \omega \in \gamma_i \in \Gamma_i.$$

Then

$$\bigcap_i K_i = M_\omega \quad \text{and} \quad K_i \left( \bigcap_{j \neq i} K_j \right) = M. \quad (1)$$

Conversely if  $\{K_i\}$  is a set of subgroups with (1) then  $\{\Gamma_i\}$  is a Cartesian decomposition of  $\Omega$  where  $\Gamma_i$  is the system of imprimitivity for  $K_i$ . Moreover,  $M_{(\{\Gamma_i\})} = M$ .

$\{K_i\}$  with (1) is called a **Cartesian system** of subgroups for  $M$ .

## Embedding Cartesian systems into larger groups

Let  $G \leq \text{Sym } \Omega$  and  $M \triangleleft G$  such that  $M$  is transitive and  $M_{(\{\Gamma_i\})} = M$ . Let  $\{K_i\}$  be the corresponding Cartesian system in  $M$ .

**Theorem**  $\{\Gamma_i\}$  is  $G$ -invariant if and only if  $\{K_i\}$  is  $G_\omega$ -invariant. The  $G_\omega$ -actions on  $\{K_i\}$  and  $\{\Gamma_i\}$  are equivalent.

$G_\omega$ -invariant Cartesian systems for  $M$



$G$ -invariant Cartesian decompositions for  $\Omega$

## Groups with a transitive minimal normal subgroup

Let  $M$  be a minimal normal subgroup of  $G$ , which is transitive and non-abelian.

Then  $M = T_1 \times \cdots \times T_k$  with  $T_1 \cong \cdots \cong T_k \cong T$  a finite simple group and  $\{T_1, \dots, T_k\}$  is a  $G$ -set.

**Theorem**  $M_{(\{\Gamma_i\})} = M$ . (So we can start looking for  $G_\omega$ -invariant Cartesian systems for  $M$ .)

**Problem:** For a fixed  $\omega$  find  $G_\omega$ -invariant  $\{K_i\}$  such that

$$\bigcap_i K_i = M_\omega \quad \text{and} \quad K_i \left( \bigcap_{j \neq i} K_j \right) = M.$$

This class of groups involves primitive groups, quasiprimitive groups, innately transitive groups with non-abelian socle.

## Normal Decompositions

$$\begin{aligned}
 M &= \boxed{T \cdots T} \quad \boxed{T \cdots T} \quad \cdots \quad \boxed{T \cdots T} \\
 K_1 &= \boxed{M_\omega \cap \square} \quad T \cdots T \quad \cdots \quad T \cdots T \\
 K_2 &= T \cdots T \quad \boxed{M_\omega \cap \square} \quad \cdots \quad T \cdots T \\
 &\vdots \\
 K_\ell &= T \cdots T \quad T \cdots T \quad \cdots \quad \boxed{M_\omega \cap \square} \\
 M_\omega &= \boxed{M_\omega \cap \square} \quad \boxed{M_\omega \cap \square} \quad \cdots \quad \boxed{M_\omega \cap \square}
 \end{aligned}$$

$\boxed{T \cdots T}, \boxed{T \cdots T}, \dots, \boxed{T \cdots T}$  is a  $G$ -invariant partition of  $\{T_1, \dots, T_k\}$ .

## Normal decompositions



$G$ -invariant partitions of  $\{T_1, \dots, T_k\}$  with

$$M_\omega = \boxed{M_\omega \cap \square} \times \boxed{M_\omega \cap \square} \times \cdots \times \boxed{M_\omega \cap \square}$$

## Transitive decompositions

Not every decomposition is normal. Let  $G = \text{PGL}_2(9)$  and consider the transitive  $G$ -action with degree 36. Then  $\text{Soc } G \cong A_6$  is transitive. By previous result  $(\text{Soc } G)_{(\{\Gamma_i\})} = \text{Soc } G$ .

There are  $K_1, K_2 \leq \text{Soc } G$  conjugate under  $G_\omega$ , such that

$$K_1 \cap K_2 = (\text{Soc } G)_\omega \text{ and } K_1 K_2 = \text{Soc } G,$$

therefore

$$[\text{Soc } G : K_1 \cap K_2] = [\text{Soc } G : K_1] \times [\text{Soc } G : K_2]$$

and  $G$  preserves this Cartesian decomposition.

**Objective:** To understand transitive decompositions for groups with a transitive minimal normal subgroup.

## Transitive decompositions

Let  $\sigma_i$  be the  $i$ -th co-ordinate projection

$$\sigma_i : (M = T_1 \times \cdots \times T_k) \rightarrow T_i.$$

Applying  $\sigma_i$  we obtain

$$\sigma_i(K_n) \left( \bigcap_{m \neq n} \sigma_i(K_m) \right) = T_i \text{ for all } i.$$

Using results about factorisations of finite simple groups (Baddeley & Praeger) we have

**Theorem** For all  $i$  there are at most three  $j$  such that  $\sigma_i(K_j) \neq T_i$ .

## Four cases to look at

- $\sigma_i(K_j) \neq T$  for three  $j$  (triple factorisations – DONE);
- $\sigma_i(K_j) \neq T$  for two  $j$  (double factorisations – HARD);
- $\sigma_i(K_j) \neq T$  for one  $j$  (normal);
- $\sigma_i(K_j) \neq T$  for zero  $j$  (normal, though not obvious).

## Transitive systems with triple factorisations

Suppose that  $j_1, j_2, j_3$  are different and

$$\underbrace{\sigma_i(K_{j_1})}_A, \underbrace{\sigma_i(K_{j_2})}_B, \underbrace{\sigma_i(K_{j_3})}_C \neq T.$$

Then

$$A(B \cap C) = B(A \cap C) = C(B \cap A) = T.$$

By Baddeley & Praeger, possibilities for  $T, A, B, C$ :  $G$ -invariant partitions

$$\{T_1, \dots, T_k\} = \bigcup_{i=1}^{\ell} \left\{ T_j \mid \sigma_j(K_i) \cong \begin{matrix} A \\ B \\ C \end{matrix} \right\}$$

## Finite simple groups with a triple factorisation

Source: Baddeley and Praeger

$T$	$A$	$B$	$C$
$\mathrm{PSp}_{4a}(2), a \geq 2$	$\mathrm{Sp}_{2a}(4).2$	$O_{4a}^-(2)$	$O_{4a}^+(2)$
$\mathrm{Sp}_6(2)$	$G_2(2)$	$O_6^-(2)$	$O_6^+(2)$
$\mathrm{Sp}_6(2)$	$G_2(2)'$	$O_6^-(2)$	$O_6^+(2)$
$\mathrm{Sp}_6(2)$	$G_2(2)$	$O_6^-(2)'$	$O_6^+(2)$
$\mathrm{Sp}_6(2)$	$G_2(2)$	$O_6^-(2)$	$O_6^+(2)'$
$\mathrm{P}\Omega_8^+(3)$	$N_1$	$P_1^\alpha$	$\mathrm{P}\Omega_8^+(2)$

## Transitive systems with triple factorisations

In all cases

$$K_i \geq \sigma_1(K_i)' \times \cdots \times \sigma_k(K_i)'.$$

When  $T \not\cong \text{Sp}_6(2)$ :

$$K_i \cong A \times \cdots \times A \times B \times \cdots \times B \times C \times \cdots \times C \times T \times \cdots \times T.$$

When  $T = \text{Sp}_6(2)$ :

$$K_i \cong \underbrace{\mathbb{Z}_2 + \cdots + \mathbb{Z}_2 + \cdots + \mathbb{Z}_2 + \cdots + \mathbb{Z}_2}_{\text{multiple copies}} \times A' \times \cdots \times A' \times B' \times \cdots \times B' \times C' \times \cdots \times C' \times T \times \cdots \times T.$$

Sufficient and necessary condition for  $\mathbb{Z}_2 + \cdots + \mathbb{Z}_2$ :

$$\underbrace{\mathbb{Z}_2 + \cdots + \mathbb{Z}_2}_{K_i} + \mathbb{Z}_2^{|\{T \leq K_i\}|} + \left( \bigcap_{j \neq i} \underbrace{\mathbb{Z}_2 + \cdots + \mathbb{Z}_2}_{K_j} + \mathbb{Z}_2^{|\{T \leq K_j\}|} \right) = \mathbb{Z}_2^k.$$

## Transitive decompositions with double factorisations

Here for all  $i$  there are only two  $j$  such that  $\sigma_i(K_j) \neq T$ . For such  $j_1, j_2$

$$\underbrace{\sigma_i(K_{j_1})}_A, \underbrace{\sigma_i(K_{j_2})}_B \neq T, \text{ such that } AB = T$$

$G$ -invariant partitions (when  $A \not\cong B$ ):

$$\begin{aligned} \{T_1, \dots, T_k\} &= \bigcup_{i=1}^{\ell} \{T_j \mid \sigma_j(K_i) \cong A\} \\ &= \bigcup_{i=1}^{\ell} \{T_j \mid \sigma_j(K_i) \cong B\}. \end{aligned}$$

Fully describing such decompositions is more difficult than finding all factorisations of finite simple groups.

## Transitive decompositions with diagonal subgroups

Suppose that there is only one  $j$  such that  $\sigma_i(K_j) \neq T_i$ . Then  $K_i$  involves a diagonal subgroup.

$$\begin{array}{rcl}
 M & = & \boxed{TT} \cdots \boxed{TT} \quad \boxed{TT} \cdots \boxed{TT} \quad \boxed{TT} \cdots \boxed{TT} \\
 K_1 & = & \color{blue}{TT} \cdots \color{blue}{TT} \quad TT \cdots TT \quad TT \cdots TT \\
 K_2 & = & TT \cdots TT \quad \color{blue}{TT} \cdots \color{blue}{TT} \quad TT \cdots TT \\
 & \vdots & \\
 K_i & = & AT \cdots TT \quad TT \cdots TT \quad TT \cdots TT \\
 & \vdots & \\
 K_j & = & TA^r \cdots TT \quad TT \cdots TT \quad TT \cdots TT \\
 & \vdots & \\
 K_\ell & = & TT \cdots TT \quad TT \cdots TT \quad \color{blue}{TT} \cdots \color{blue}{TT}
 \end{array}$$

$G$ -invariant partitions:

$$\begin{aligned}
 \{T_1, \dots, T_k\} &= \boxed{TT} \dot{\cup} \cdots \dot{\cup} \boxed{TT} = \color{red}{\square} \dot{\cup} \cdots \dot{\cup} \color{red}{\square} \\
 &= \bigcup_{j=1}^{\ell} \{T_i \mid \sigma_i(K_j) \cong A\}
 \end{aligned}$$

## Transitive decompositions with diagonal subgroups

We have  $AA^T = T$ . Possibilities for  $A$  and  $T$  (Baddeley & Praeger):

$T$	$A$
$A_6$	$A_5$
$M_{12}$	$M_{11}$
$P\Omega_8^+(q), q \geq 3$	$\Omega_7(q)$
$P\Omega_8^+(2)$	$PSp_6(2)$
$PSp_4(q), q \geq 4$ even	$Sp_2(q^2).2$

## Transitive decompositions with diagonal subgroups

Note:  $A$  is almost simple. Moreover

$$K_i \geqslant TT \times \cdots \times TT \times A' \times \cdots \times A' \times T \times \cdots \times T.$$

When  $A$  is simple:

$$K_i = TT \times \cdots \times TT \times A \times \cdots \times A \times T \times \cdots \times T$$

When  $A$  is not simple:

$$K_i = TT \times \cdots \times TT \times \begin{matrix} \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \\ A' \times \cdots \times A' \end{matrix} \times T \times \cdots \times T$$

Sufficient and necessary condition for  $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ .