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A system theoretic approach to behavioral finance

László Gerencsér and Zalán Mátyás

Abstract— **The purpose of this paper is to study a particular example of a feedback system modelling the behavior of agents in financial markets. The agent's decision is based on his beliefs of the price dynamics and his behavior reflecting his attitude, such as risk aversion or risk preference. The convergence of the resulting iterative procedure is examined. A data driven stochastic approximation procedure for estimating the price dynamics is suggested. Simulation results for various behaviors are also presented.**

I. INTRODUCTION

Financial markets are commonly modeled as open-loop systems where a causal relationship between certain variables such as the demand for a particular stock and its price is assumed. However, the discrimination between input and output is often arbitrary, as pointed out by Willems [15]. This mutual causal dependence gives rise to a feedback system.

The purpose of this paper is to study a particular example of a feedback system modeling the behavior of agents of financial markets. The agent (whose effect on the market is assumed to be macroeconomically significant) predicts the observed price process, and using these predictions will buy or sell shares according to his/her strategy or behavior which reflects his/her risk aversion, conservatism, etc. A variety of behaviors of economic players is described in Shefrin [13] and Kostolany [10]. The agent's action will then be collected by the market (together with some noise) and thus we get a closed loop system.

A key factor in the above model is the agent's belief of the price dynamics, and his/her predictive capability. For any fixed predictor of the price process, denoted by M, we get a closed loop dynamics and a price dynamics depending on M, for which the optimal predictor will be typically different from M. It is then reasonable to remodel the price process and use a new, better predictor. This iterative procedure will be described and analyzed for linear systems in term of transfer functions. An on-line, data-driven procedure will also be presented.

II. A BEHAVIORAL MODEL

The price of a given stock at time $t \in \mathbb{Z}_+$ is denoted by p_t. If we consider a portfolio with n stocks then the above defined terms are multi-dimensional vectors in $\mathbb{R}^n$, their i^{th} component expressing the price of the i^{th} stock. Naturally, all components of the price process (p_t) should be non-negative. However, if we think of the price of a stock as a measure of the profitability of the company that issues it, then a negative price could mean that the company is unprofitable. Therefore, we assume that the price process p of a stock can take any value in $\mathbb{R}$.

The prices are given by the market. In general, the stock market is modeled as a dynamical system that generates prices based on the transaction requests of the agents. Typically, the prices of the next period are calculated by an automated trading system that uses an equilibrium-price transaction matching algorithm (for a currently used algorithm at the Budapest Stock Exchange see [2]). In this paper the market is viewed as a black box, denoted by P, relating the input process u (containing the transaction requests) to the output process p:

$$Pu = p \ . \tag{1}$$

It is assumed that stock prices depend on the past and present values of the demand process and on the current stochastic disturbances.

The agent relates the price and the demand processes as

$$d = -Cp \tag{2}$$

where C is a strictly causal dynamical system representing the strategy of the agent. The components of the demand sequence (d_t) can also take any values in $\mathbb{R}$: this time a negative value means that the agent would like to sell the stock concerned. We assume that stocks are infinitely divisible: any amount can be purchased or sold.

The interconnection of the two systems is given by the equation

$$d_t + e_t = u_t \tag{3}$$

where the stochastic disturbance (e_t) is a stationary process.

Remark. The way randomness should be included in the model is by no means an obvious thing. Using an additive exogenous white noise at the input node of the market turns our model into the familiar form of a stochastic feedback system, the plant being the stock market and the agent's behavior being the controller. The resulting stochastic feedback system is shown in Fig. 1.

Now assume that a new agent enters the stock market. He observes a stock price process p which he assumes to

László Gerencsér and Zalán Mátyás are with the Computer and Automation Institute of the Hungarian Academy of Sciences, MTA SZTAKI, 13-17 Kende u., Budapest 1111, Hungary; gerencser@sztaki.hu, matyas@sztaki.hu

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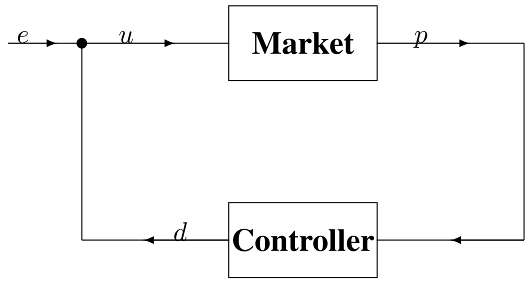

Fig. 1. The closed-loop system.

be stationary. Then based on his beliefs and on other side information, he constructs a price predictor M:

$$\hat{p} = Mp \tag{4}$$

where M is assumed to be a *strictly non-anticipating* predictor, meaning that $\hat{p}_t$ depends only on the values p_s for $s < t$.

The agent uses this predicted price to determine his own demand. We allow the possibility that he/she behaves less than fully rationally. Proponents of behavioral finance (as the set of theories based on this assumption is usually referred to) argue that psychological phenomena prevent decision makers from acting in a rational manner (see for example Greenfinch [6] and Shefrin [13]). Critics of this theory (see the works of Lucas [12] and Simon [14]) claim that the behavior of the agents is always rational from a particular perspective. In any case the strategy of the agent can be formalized as

$$B\begin{pmatrix} p \\ \hat{p} \end{pmatrix} = d \tag{5}$$

where B is assumed to be a strictly non-anticipating operator. The demand at a given time depends only on the past values of the price process. In this paper we assume that the behavior of the agent is fixed, i.e. the operator B is considered to be given.

Combining equations (1) - (5), we get the following diagram.

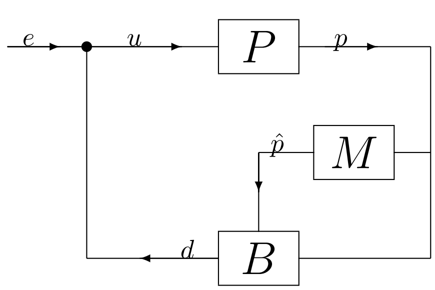

Fig. 2. A behavioral feedback model.

Examples. A psychological phenomenon extensively studied by behaviorists is the so-called *loss aversion*. Nobel prize winner psychologists Kahnemann and Tversky [9] find that even simple risk aversion can be biased: empirical evidence shows that a loss has about two and a half times the impact of a gain of the same magnitude. This behavior can be formalized by the equation

$$d_t - \alpha d_{t-1} = N(\hat{p}_t - p_{t-1})^+ - 0.4N(\hat{p}_t - p_{t-1})^-$$

where N is the number of stocks the agent wants to purchase and $0 < \alpha < 1$, $\alpha \approx 1$ is a parameter expressing the faith of the agent in his past decisions.
A less sophisticated "rational" behavior is described by the equation

$$d_t - \alpha d_{t-1} = N \ \mathrm{sgn}(\hat{p}_t - p_{t-1})$$

where the notations of the previous example are used.

Now imagine how an agent using the above scheme would determine his demand for a particular stock. Fixing $M = M_0$ (together with the behavior B) the controller $C = C(M_0)$ can be determined. Assuming that the closed loop system is well-defined, the resulting price process $p = p(M_0)$ will be a stationary process with a spectrum depending on C. The agent observes this process and calculates its least squares predictor M^+, which will also depend on C, say $M^+ = M^+(C(M_0))$. Next the agent compares the two predictors. If $M^+(C(M_0)) \neq M_0$ then it is reasonable to switch to this better predictor: he simply puts $M_1 := M^+(C(M_0))$ and uses this new predictor when determining his demand.

As long as $M^+(C(M_i)) \neq M_i$, the agent repeatedly updates his predictor. The following questions arise:

- Let $f(M) = M^+(C(M))$. Does the iterative procedure
$$M_{i+1} = f(M_i) \tag{6}$$
converge?
- Is there a predictor M^* for which the market is in equilibrium, i.e. M^* is a fixed point of the operator equation $M^* = f(M^*)$?
- Let M be estimated on-line from observed values of p. Does the resulting stochastic approximation procedure converge?

To have an idea of the mechanism of prediction-based behavioral models, from now on we assume that the dynamical systems P, M, B defined above are all linear. In particular P is a rational non-anticipating operator with an invertible constant term P_0, M is strictly non-anticipating (having no direct term) and

$$B\begin{pmatrix} p \\ \hat{p} \end{pmatrix} = B_1 p + B_2 \hat{p} \tag{7}$$

where B_1 is a strictly non-anticipating and B_2 is a non-anticipating linear operator.

Assuming that P is minimum phase, the optimal one-step ahead predictor of p is known to be given by

$$\hat{p} = (I - P_0 H^{-1})p \ ,$$

where H denotes the transfer function from e to p (see Caines [3] and Hannan and Deistler [7]). Assuming that $||P_0 B_2||_{H_\infty} < 1$, easy calculation yields that the iterating procedure converges to a uniquely determined predictor given by

$$M^* = (I - P_0 B_2)^{-1}(I - P_0 P^{-1} + P_0 B_1) \ . \quad (8)$$

In the general case P may be non-minimum phase since there is a delay in the market's response to change in demands. Then we first have to perform spectral factorization of H. Denoting the stable and minimum-phase spectral factor by L, the least squares predictor is obtained by

$$\hat{p} = (I - L^{-1})p \ ,$$

and thus $f(M)$ is defined by the following sequence of equations:

$$\begin{aligned} C &= -(B_1 + B_2 M) \\ H &= (I + PC)^{-1} P \\ L\bar{L}^T &= H P_0^{-1} (\bar{P}_0^{-1})^T \bar{H}^T \\ f(M) &= I - L^{-1} \ . \end{aligned}$$

Conditions under which the above mapping f is a contraction are yet to be developed.

III. A DATA-DRIVEN PROCEDURE

The updating of M is easy if P is rational and minimum phase: all we need is to identify P and use (8). The situation is completely different in real life financial markets where P is generally non-rational and non-minimum phase. It is then more reasonable to identify M^* directly from the data since to determine the predicted price we would need to perform spectral factorization. Write

$$p = L\nu$$

where ν is the innovation process of p. In order to approximate L on the basis of observed values of p, we choose the best approximation from some parametric family $\mathcal{A} := \{A(\theta) : \theta \in D \subseteq \mathbb{R}^k\}$. If we have little prior information, we can fit an $AR(k)$ model to our data, i.e. we may try to use low order predictors. In this case

$$\mathcal{A}_k = \{A \mid \deg A \leq k, \ A_0 = I, \ A \text{ stable}\}$$

where A is a polynomial of the shift operator and A_0 is its leading coefficient. The best kth-order estimator of the system is defined by least squares fit: minimize $E|Ap|^2$ subject to $A \in \mathcal{A}_k$. The coefficients of the optimal solution can be estimated by solving the minimization problem

$$\min_{A \in \mathcal{A}_k} \sum_{n=1}^{N} (Ap)^2 \ ,$$

which is quadratic in the coefficients of A and thus can easily be computed. Denoting the solution by A_k^*, the predicted price process is defined by $\hat{p} = (I - A_k^*)p$.

The iterative procedure now takes the following form. Let us consider a parametric family of predictors M corresponding to an $AR(k)$-model of p. If the $AR(k)$-model is parametrized by η then we have

$$M(\eta) = I - A(\eta).$$

Let us now fix a predictor, or equivalently fix an η. Closing the loop results in the price process $p(\eta)$. We fit an $AR(k)$ model to $p(\eta)$ by a least squares fit which is equivalent to minimizing $E|A(\theta)p(\eta)|^2$ subject to $A(\theta) \in \mathcal{A}_k$. Using the notation

$$\nu(\theta, \eta) := A(\theta)p(\eta)$$

and

$$W(\theta, \eta) := \frac{1}{2} E|\nu(\theta, \eta)|^2$$

we have to solve the linear equation

$$W_\theta(\theta, \eta) = 0. \quad (9)$$

The solution will be denoted by $\varphi(\eta)$. It is then reasonable to use this best $AR(k)$ fit for prediction in the next phase, i.e. we redefine M as

$$M^+ = I - A(\varphi(\eta))$$

Thus the mapping

$$M^+ = f(M)$$

defined above in terms of transfer functions will be reduced to a mapping

$$\eta^+ = \varphi(\eta).$$

Market equilibrium is achieved if

$$\eta = \varphi(\eta),$$

i.e. we would like to solve

$$W_\theta(\eta, \eta) = 0.$$

Let the solution be denoted by η^*. Introduce

$$G(\eta) := W_\theta(\eta, \eta) = E\nu_\theta(\eta, \eta)\nu(\eta, \eta).$$

Then η^* is simply the solution of

$$G(\eta) = 0. \quad (10)$$

Since $W_\theta(\theta, \eta)$ is computable experimentally for each θ and η, therefore the general stochastic approximation procedure developed in Benveniste et al. [1], Ljung and Söderström [11] is applicable. We propose the following data-driven procedure to solve (10):

$$\eta_{n+1} = \eta_n - \frac{c}{n} \nu_{\theta n} \nu_n \quad (11)$$

where ν_n, $\nu_{\theta n}$ are on-line estimates of $\nu(\eta_n, \eta_n)$ and $\nu_\theta(\eta_n, \eta_n)$, respectively, and $c > 0$ is some step size. Taking into account the definition of $\nu(\theta, \eta)$ we have

$$\nu_{\theta n} = (p_{n-1}, \ldots, p_{n-k})$$

and

$$\nu_n = [A(\eta_n)p]_n.$$

The associated ODE (see Benveniste et al. [1], Gerencsér [4]) belonging to the above stochastic approximation is given by

$$\dot{\eta}_s = -c\ G(\eta_s)\ . \tag{12}$$

The Jacobian-matrix of G is

$$G_\eta(\eta) = W_{\theta\theta}(\eta,\eta) + W_{\theta\eta}(\eta,\eta).$$

We know that $W_{\theta\theta}(\theta,\eta)$ is positive semidefinite for each θ and η, and in fact it is independent of θ for fixed η. Note, however, that we have no control of the second term $W_{\theta\eta}(\eta,\eta)$. Therefore the asymptotic stability of (12) is not guaranteed.

One simple way would be to resort to the use of a Newton-method. Unfortunately the Jacobian $G_\theta(\theta)$ can not be computed experimentally, since $W_{\theta\eta}(\eta,\eta)$ contains the derivative $(\partial/\partial\eta)p(\eta)$. It is well known (see Hjalmarsson [8]) that in certain feedback systems, the derivative of the output process with respect to the parameter of the controller can be calculated by means of experimenting with the original system. This, however, is unfeasible at present.

IV. Simulation results

The data-driven procedure mentioned above was simulated for various behaviors (loss aversion, rational behavior and risk seeking behavior) in a MATLAB environment. The market dynamics was taken to be

$$p_t = 0.8p_{t-1} - 0.6p_{t-2} + 2.4(d_t - d_{t-1}) + e_t$$

and we approximated the dynamics of p_t by an $AR(2)$ model.

Fig. 3 shows the evolution of the parameter process $\eta_n = (\eta_n^{(1)}, \eta_n^{(2)})^T$ for the phenomenon of loss aversion:

$$d_t - d_{t-1} = (\hat{p}_t - p_{t-1})^+ - 0.4(\hat{p}_t - p_{t-1})^-.$$

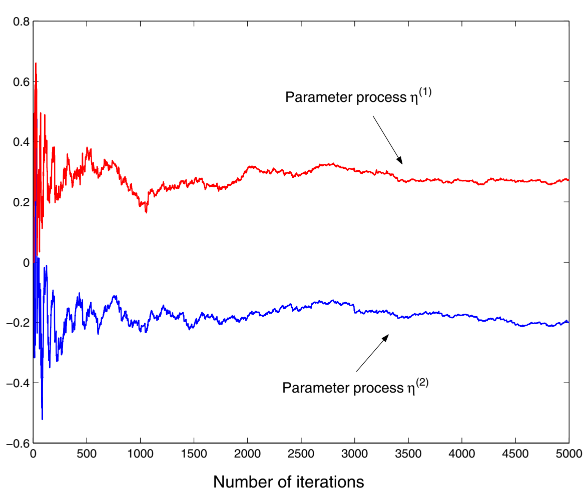

Fig. 3. Loss aversion

The same market dynamics and the rational behavior

$$d_t - d_{t-1} = \text{sgn}(\hat{p}_t - p_{t-1})$$

yields Fig. 4. .

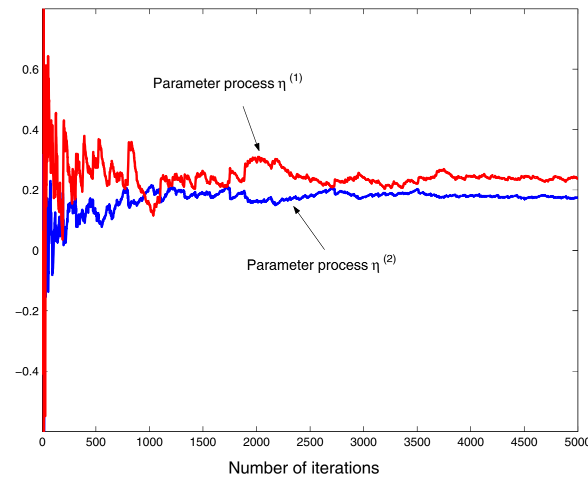

Fig. 4. Rational behavior

In case of the risk seeking behavior

$$d_t - d_{t-1} = \begin{cases} 1 & \text{if } \hat{p}_t > 0.95\ p_{t-1} \\ -1 & \text{if } \hat{p}_t \le 0.95\ p_{t-1} \end{cases}$$

the resulting parameter process does not converge.

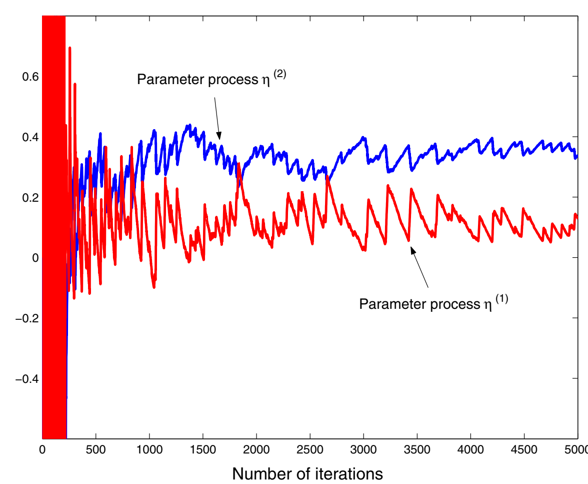

Fig. 5. Risk seeking behavior

V. Conclusion

A behavioral finance model was considered from the viewpoint of systems and control theory. Iterative procedures for finding market equilibrium have been proposed and analyzed for linear systems. Determining the stability of the associated ODE remains an open problem. Simulation results for several behaviors were also presented.

The presented model could serve as a basis for a comprehensive stock market model. In such a multi-player model, all the participating investors should be connected parallelly to the plant. The different behaviors of the agents should determine not only the demand for the stocks but also the bid prices at which agents are willing to make transactions (short selling and budget constraints could also be included). These features (and many other) can easily be incorporated into the model presented.

VI. ACKNOWLEDGEMENT

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