

A minimax problem for set systems

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Summary

Let $[2k]$ denote the set $\{1, 2, \dots, 2k\}$, $\binom{A}{k}$ the system of all k -element subsets of a set $A \subseteq [2k]$. Consider the minimax problem

$$f_k = \min_{S \in \mathcal{F}_k} \max_{B \in \binom{[2k]}{k+1}} \left| S \cap \binom{B}{k} \right| = ?$$

where $\mathcal{F}_k = \left\{ S : S \cup \bar{S} = \binom{[2k]}{k} \right\}$ and $\bar{S} = \{[2k] \setminus A : A \in S\}$. It is shown in the paper that $f_1 = 1, f_2 = f_3 = 2, f_4 = 3$ and $f_5 = 4$. In general, a lower and an upper bound on f_k are given as follows: $(k+1)/2 \leq f_k \leq k$ where equality regarding the lower bound can hold only if $k = 2^t - 1$ ($t = 1, 2, 3, \dots$). For k large, the upper bound is improved to $f_k \leq (k+1)/2 + O(\sqrt{k \ln k})$, yielding a tight asymptotic estimate as $k \rightarrow \infty$.