

Fast Quantum Algorithms

Lectures 1 and 2

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 - QFT over abelian groups
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 - HSP in lattices
 - Units in number fields and hidden lattices
 - Open problems

Simon's problem

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 - Remark: reducible to the decision version
- Classically difficult:
with $2^{\frac{n}{4}}$ queries can guess the case only with probability
 $\leq \frac{1}{2} + \frac{1}{2^{n/2}}$

Simon's algorithm

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Hadamard $^{\otimes n}$

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measure $f(x)$, drop it

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- $|x\rangle + |x + u\rangle$

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 - $\sum_{x \in \mathbb{Z}_2^n} |x\rangle|f(x)\rangle$
 - ↓
 - $|x\rangle + |x + u\rangle$
 - ↓
 - $\sum_{y \in \mathbb{Z}_2^n} ((-1)^{(x,y)} + (-1)^{(x,y)+(u,y)}) |y\rangle$
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Simon's algorithm 2

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• random $y \in u^\perp$
- measure

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- $\ell = O(n)$ iteration gives y_1, \dots, y_ℓ :

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 uses Grover's techniques

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QFT mod 2^ℓ

- Fourier transform mod 2^ℓ :

$$\Phi_{2^\ell} : |j\rangle \mapsto \sum_{k=0}^{2^\ell-1} \omega^{kj} |k\rangle,$$

$$\text{where } \omega = \sqrt[2^\ell]{1} \left(= e^{\frac{2\pi i}{2^\ell}} \right)$$

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standard basis \rightarrow eigenvectors of shift mod 2^n .

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$$|j\rangle = |j_{\ell-1}\rangle |j_{\ell-2}\rangle \dots |j_1\rangle |j_0\rangle,$$

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- induction:

$$|j\rangle = |[j/2]\rangle |j_0\rangle$$

QFT mod 2^ℓ , part 2

$$\Phi_{2^\ell} |j\rangle = \sum_{k'=0}^{2^\ell-1} \omega^{k'j} |k'\rangle$$

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$$\begin{aligned}\Phi_{2^\ell}|j\rangle &= \sum_{k'=0}^{2^\ell-1} \omega^{k'j}|k'\rangle \\ &= \sum_{k=0}^{2^{\ell-1}-1} \omega^{2kj}|2k\rangle + \sum_{k=0}^{2^{\ell-1}-1} \omega^{(2k+1)j}|2k+1\rangle\end{aligned}$$

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QFT $_2^{\ell-1}$ on $|[j/2]\rangle$

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- Procedure:

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- $|j\rangle \otimes (\Phi_{2^{\ell-1}} |j/2\rangle) \otimes |0\rangle$

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- $|j\rangle \otimes (\Phi_{2^{\ell-1}}|[j/2]\rangle) \otimes (|0\rangle + |1\rangle)$

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for $(t \in [0, \ell - 1])$

if $(j_t \neq 0)$ then do cond. phase shift

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↓

for $(t \in [0, \ell - 1])$

if $(j_t \neq 0)$ then do cond. phase shift

- $|j\rangle \otimes \Phi_{2^\ell}(|j\rangle)$

QFT mod 2^ℓ - simple implementation 2

- So far: $P : |j\rangle \otimes |0\rangle \mapsto |j\rangle \otimes \Phi(|j\rangle)$

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- Simple implementation of $|j\rangle \mapsto \otimes \Phi(j)$ (using aux qbits):

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- Simple implementation of $|j\rangle \mapsto \otimes \Phi(j)$ (using aux qbits):
 - $|j\rangle \otimes |0\rangle$

QFT mod 2^l - simple implementation 2

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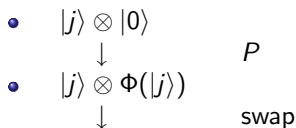
- $$\begin{array}{ccc} |j\rangle \otimes |0\rangle & & \\ \downarrow & & P \end{array}$$

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↓
• $|j\rangle \otimes \Phi(|j\rangle)$ P

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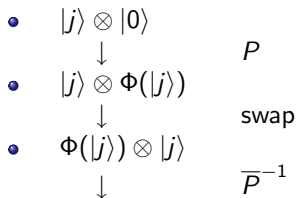


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- Simple implementation of $|j\rangle \mapsto \otimes \Phi(j)$ (using aux qbits):
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↓ P
 - $|j\rangle \otimes \Phi(|j\rangle)$
↓ swap
 - $\Phi(|j\rangle) \otimes |j\rangle$

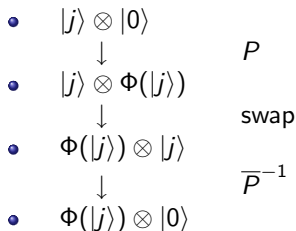
QFT mod 2^ℓ - simple implementation 2

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Details in *Cleve, Ekert, Macchiavello, Mosca (1998)*

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- Details: *In: e.g., Cleve, Ekert, Macchiavello, Mosca (1998).*

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- Gadget: quantum graph (diagram) of f :

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Computing $|f_N\rangle$ for $N = 2^\ell$:

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Decomposition of $|f_N\rangle$:

$$|f_N\rangle = \sum_{x=0}^{N-1} |x\rangle |f(x)\rangle \approx \sum_{y=0}^{r-1} \left(\sum_{z=0}^{\lfloor \frac{N}{r} \rfloor - 1} |rz + y\rangle \right) |f(y)\rangle$$

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- Decompose into eigenvectors of shift mod $r\lfloor \frac{N}{r} \rfloor$ (QFT "in mind"):

$$\sum_{j=0}^{r\lfloor \frac{N}{r} \rfloor - 1} c_{yj} u_j, \quad \text{where } u_j = \sum_{k=0}^{r\lfloor \frac{N}{r} \rfloor - 1} \omega^{-kj} |k\rangle.$$

Period finding 4

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$$\omega^{jy} \begin{cases} \lfloor \frac{N}{r} \rfloor & \text{if } j \equiv 0 \pmod{\lfloor \frac{N}{r} \rfloor} \\ 0 & \text{otherwise} \end{cases}$$

where $\omega = \sqrt[r^{\lfloor N/r \rfloor}]{1}$.

Period finding 5

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$$\omega^{\ell\lceil N/r \rceil} = \left(\sqrt[r\lceil N/r \rceil]{1} \right)^{\ell\lceil N/r \rceil} = \left(\sqrt[r]{1} \right)^{\ell}$$

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- Original: *Shor 1994*

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Contents

- 1 Simon's algorithm
- 2 Basic tools
 - QFT mod powers of 2
 - Phase estimation
 - Period finding
 - QFT over abelian groups
- 3 The HSP
 - The Hidden Subgroup Problem
 - Coset states
 - Abelian Fourier sampling
 - Applications of abelian HSP
- 4 Infinite abelian HSPs
 - HSP in lattices
 - Units in number fields and hidden lattices
 - Open problems

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Discrete log $G = \mathbb{Z}_n \times \mathbb{Z}_n$, $f(k, \ell) = u^k v^{-\ell}$,
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 - $\Gamma_1 \cong \Gamma_2$ iff

$$\left| Aut(\Gamma_1 \dot{\cup} \Gamma_2) \right| = 2 \cdot |Aut(\Gamma_1)| \cdot |Aut(\Gamma_2)|$$

Coset states

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T : left transversal of H

= a set of left coset representatives by H

Coset states 2

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Coset states - summary

Coset state (with random $a \in T$ (random $a \in G$))

$$|aH\rangle = \frac{1}{\sqrt{|H|}} \sum_{x \in H} |ax\rangle$$

(normalizing factor included)

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- $\sum_{\chi \in \hat{G}} (\chi(a) \sum_{x \in H} \chi(x)) |\chi\rangle$
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- computing H : system of linear congruences.

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Classically difficult, even in the exponent 2 case

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- Probably normal HSP in other cases (Ákos)

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Contents

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- 2 Basic tools
 - QFT mod powers of 2
 - Phase estimation
 - Period finding
 - QFT over abelian groups
- 3 The HSP
 - The Hidden Subgroup Problem
 - Coset states
 - Abelian Fourier sampling
 - Applications of abelian HSP
- 4 Infinite abelian HSPs
 - HSP in lattices
 - Units in number fields and hidden lattices
 - Open problems

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- Remarks: for $y \in \mathcal{O}^*$:

$$\prod_{i=1}^{r_1} |y|_i \prod_{j=r_1+1}^{r_1+r_2} |y|_j^2 = \text{Norm}(y) = 1.$$

$$\mathcal{O} \cap \ker \text{Log} = \langle \sqrt[r]{1} \rangle \text{ (Kronecker)}$$

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compact repr. of generators for \mathcal{O}^* (Thiel).

Principal ideal

- I principal, if $I = a\mathcal{O}$ for some $a \in K^*$
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- Task: given I (by HNF), find a s.t. $I = a\mathcal{O}$
(or "not principal")
- Discrete log-like hiding function:
 - $F_I : \mathbb{Z} \times \mathbb{R}^r$
 - assume $I = a\mathcal{O}$, $I = I_\zeta$ with $\zeta = \text{Log}(a)$
 - want: $F_I(k, x) = F(I_{k\zeta - x}) = (I_{k\zeta - x}, \delta_{k\zeta - x})$
 - $k\zeta - x = \text{Log}(\text{minimum of } I^k)$ "downwards closest" to $-x$
 - this computes F_I without knowing ζ
- Hidden subgroup $\ni (1, \zeta')$ $\zeta' = \text{Log}(a')$, $I = a\mathcal{O}$.

Class group under GRH

- Thiel (94): "small" prime ideals P_1, \dots, P_ℓ generate class group
- $G = \mathbb{Z}^\ell$, (quantum-valued) hiding function:

$$(k_1, \dots, k_\ell) \mapsto |R(J)\rangle = \sum_{I \in R(J)} |I\rangle, \text{ where}$$

- $J = P_1^{k_1} \dots P_\ell^{k_\ell}$,
- $R(J) = \{\text{reduced ideals } \sim J\}$
- computing $|J\rangle|0\rangle \mapsto |J\rangle|R(J)\rangle$:
 - $M(J) := \{\text{minima of } J\}$
 - "easy": $|J\rangle \sum_{\mu \in M(J)} |\mu\rangle |\mu^{-1}J\rangle$
 - $|J\rangle \sum_{I \in R(J)} \left(\sum_{\mu \in M(J): I = \mu^{-1}J} |\mu\rangle |J\rangle \right) |I\rangle$
 - term in middle term computable from I and J
(principal ideal algorithm)

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